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Algebraic Surfaces with Automorphisms

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Introduction

In my thesis I worked on two different projects, both related with projective surfaces with automorphisms. In the first one I studied Abelian surfaces with an automorphism and quaternionic multiplication: this work has already been accepted for publication in the Canadian Journal of Mathematics (see [BvG15]). In the second project I treat surfaces isogenous to a product of curves and their cohomology.

Abelian Surfaces with an Automorphism

The Abelian surfaces, with a polarization of a fixed type, whose endomorphism ring is an order in a quaternion algebra are parametrized by a curve, called Shimura curve, in the moduli space of polarized Abelian surfaces. There have been several attempts to find concrete examples of such Shimura curves and of the Abelian surfaces over this curve. In [HM95] Hashimoto and Murabayashi find Shimura curves as the intersection, in the moduli space of principally polarized Abelian surfaces, of Humbert surfaces. Such Humbert surfaces are now known “explicitly” in many other cases and this might allow one to find explicit models of other Shimura curves. Other approaches are taken in [Elk08] and [PS11].

We consider the rather special case where one of the Abelian surfaces in the family is the selfproduct of an elliptic curve. We assume this elliptic curve to have an automorphism of order three or four. For a fixed product polarization of type $(1, d)$, we denote by $\mathbb{H}_{j,d}$ the set of the deformations of the selfproduct with the automorphism of order j . We prove the following theorem:

Theorem 1.2.4. *Let $j \in \{3, 4\}$, $d \in \mathbb{Z}$, $d > 0$ and let $\tau \in \mathbb{H}_{j,d}$, so that the Abelian surface $A_{\tau,d}$ has an automorphism ϕ_j of order j . Then the endomorphism algebra of $A_{\tau,d}$ also contains an element ψ_j with $\psi_j^2 = d$. Moreover, for a general $\tau \in \mathbb{H}_{j,d}$ one has*

$$\mathrm{End}(A_{\tau,d})_{\mathbb{Q}} \cong \frac{(-j, d)}{\mathbb{Q}}.$$

where $(a, b)/\mathbb{Q} := \mathbb{Q}1 \oplus \mathbb{Q}\mathbf{i} \oplus \mathbb{Q}\mathbf{j} \oplus \mathbb{Q}\mathbf{k}$ is the quaternion algebra with $\mathbf{i}^2 = a$, $\mathbf{j}^2 = b$ and $\mathbf{ij} = -\mathbf{ji}$.

It is easy to compute for which d the quaternion algebra $(-j, d)/\mathbb{Q}$ is a skew field: for these d the general Abelian surface in the family $\mathbb{H}_{j,d}$ is simple. In particular this provides examples of simple Abelian surfaces with an automorphism of order three and four. This construction, together with well-known results about automorphisms of Abelian surfaces (see [BL04, Chapter 13]), leads to:

Theorem 1.2.7. *Let A be a simple Abelian surface and $\varphi \in \text{Aut}(A)$ a non-trivial automorphism (i.e. $\varphi \neq \pm 1_A$) of finite order. Then $\text{ord}(\varphi) \in \{3, 4, 5, 6, 10\}$.*

We focus in particular on the family $\mathbb{H}_{3,2}$ of Abelian surfaces with an automorphism of order three and a polarization of type $(1, 2)$. In [Bar87] Barth provides a description of a moduli space $M_{2,4}$, embedded in \mathbb{P}^5 , of $(2, 4)$ -polarized Abelian surfaces with a level structure. Since the polarized Abelian surfaces we consider have an automorphism of order three, the corresponding points in $M_{2,4}$ are fixed by an automorphism of order three of \mathbb{P}^5 . This allows us to explicitly identify the Shimura curve in $M_{2,4}$ that parametrizes the Abelian surfaces with quaternionic multiplication by the maximal order \mathcal{O}_6 in the quaternion algebra with discriminant 6. It is embedded as a line in $M_{2,4} \subset \mathbb{P}^5$ and the symmetric group \mathcal{S}_4 acts on this line by changing the level structures. According to Rotger [Rot04], an Abelian surface with endomorphism ring \mathcal{O}_6 is the Jacobian of a unique genus two curve. We show explicitly how to find such genus two curves, or rather their images in the Kummer surface embedded in \mathbb{P}^5 with a $(2, 4)$ -polarization. These curves were already been considered by Hashimoto and Murabayashi in [HM95]: we give the explicit relation between their description and ours. Moreover we find a Humbert surface in $M_{2,4}$ that parametrizes Abelian surfaces with $\mathbb{Z}(\sqrt{2})$ in the endomorphism ring.

Cohomology of surfaces isogenous to a product

Surfaces isogenous to a product of curves provide examples of surfaces of general type with many different geometrical invariants. They have been introduced by Catanese in [Cat00]:

Definition. A smooth surface S is said to be isogenous to a product (of curves) if it is isomorphic to a quotient $\frac{C \times D}{G}$ where C and D are curves of genus at least one and G is a finite group acting freely on $C \times D$.

We say that a surface isogenous to a product is of mixed type if there exists an element of G interchanging the two curves; otherwise, if G acts

diagonally on the product, we say that the surface is of unmixed type. A surface isogenous to a product is of general type if the genus of both curves, C and D , is greater or equal to 2: in this case we say that the surface is isogenous to a higher product.

The cohomology groups of a surface $S = (C \times D)/G$ isogenous to a product of unmixed type are determined by the action of the group G on the cohomology groups of the curves. Moreover the action of an automorphism group G on a smooth curve C forces a decomposition of the first cohomology group, as described in [BL04, section 13.6] and in [Roj07]:

Proposition (Group algebra decomposition). *Let G be a finite group acting on a curve C . Let W_1, \dots, W_r denote the irreducible rational representations of G and let $n_i := \dim_{D_i}(W_i)$, with $D_i := \text{End}_G(W_i)$, for $i = 1, \dots, r$. Then there are rational Hodge substructures B_1, \dots, B_r such that*

$$H^1(C, \mathbb{Q}) \simeq \bigoplus_{i=1}^r n_i B_i.$$

From this a decomposition of the cohomology groups of the surface S follows directly. We apply these results to surfaces isogenous to a higher product of unmixed type with $\chi(\mathcal{O}_S) = 2$ and $q(S) = 0$: they have been studied and classified by Gleissner in [Gle13]. For these surfaces the Hodge diamond is fixed and in particular the Hodge numbers of the second cohomology groups are the same as those of an Abelian surface:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 4 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

From Gleissner's classification we obtain a complete list of the 21 possible groups G . We proved that the second cohomology group of these surfaces can be described explicitly as follows:

Theorem 2.5.1. *Let S be a surface isogenous to a higher product of unmixed type with $\chi(\mathcal{O}_S) = 2$, $q(S) = 0$. Then there exist two elliptic curves E_1 and E_2 such that $H^2(S, \mathbb{Q}) \cong H^2(E_1 \times E_2, \mathbb{Q})$ as rational Hodge structures.*

In general it is not possible to construct these elliptic curves “geometrically” using the action of G . More precisely there are no intermediate coverings $\pi_F : C \rightarrow C/F$ and $\pi_H : D \rightarrow D/H$, $F, H \leq G$ with $C/F \cong E_1$ and $D/H \cong E_2$: we can only prove that such elliptic curves must exist. The proof of the theorem is standard for all but four groups: in these cases we study one by one the corresponding surfaces in order to construct the elliptic curves. As a further application we use this approach to study some surfaces isogenous to a higher product with $p_g = q = 2$, in particular those are of Albanese general type.

Chapter 1

Abelian surfaces with an automorphism

1.1 Basic definitions

Here we recall some basic notions about Abelian varieties and endomorphisms of Abelian varieties. For a complete reference see [BL04] or [Dol14].

1.1.1 Complex tori and Abelian varieties

Definition 1.1.1. A complex torus T of dimension g is a quotient $T = V/\Lambda$ where V is a complex vector space of dimension g and Λ is a lattice in V . An Abelian variety is a projective complex torus.

By Kodaira Embedding Theorem (see [GH94, pag. 191]) a complex torus is projective if and only if it admits an ample line bundle.

Definition 1.1.2. Let A be an Abelian variety. A polarization on A is the first Chern class of an ample line bundle L .

We refer to the pair $(A, c_1(L))$ as a polarized Abelian variety.

Let $T = V/\Lambda$ be a complex torus of dimension g . We set $\Lambda_{\mathbb{R}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. There is a natural identification between $H^2(T, \mathbb{Z})$ and bilinear alternating forms $E : \Lambda_{\mathbb{R}} \times \Lambda_{\mathbb{R}} \rightarrow \mathbb{R}$ such that $E(\Lambda, \Lambda) \subseteq \mathbb{Z}$. We identify $\Lambda_{\mathbb{R}}$ with V as real vector spaces of dimension $2g$: the multiplication by i on V induces an almost complex structure $J : \Lambda_{\mathbb{R}} \rightarrow \Lambda_{\mathbb{R}}$ such that $J^2 = -Id$.

Proposition 1.1.3 (Riemann Conditions). *For an alternating form $E : \Lambda_{\mathbb{R}} \times \Lambda_{\mathbb{R}} \rightarrow \mathbb{R}$ the following conditions are equivalent:*

- *There is an ample line bundle L on T such that E represents the first Chern class $c_1(L)$, via the identification above.*
- *$E(\Lambda, \Lambda) \subseteq \mathbb{Z}$, $E(Jx, Jy) = E(x, y)$ for all $x, y \in \Lambda_{\mathbb{R}}$ and $E(Jx, x) > 0$ for all $x \in \Lambda_{\mathbb{R}}$, $x \neq 0$.*

Lemma 1.1.4 (Frobenius Lemma). *If $E : \Lambda \times \Lambda \rightarrow \mathbb{Z}$ is a bilinear alternating form on $\Lambda \cong \mathbb{Z}^{2g}$ then there exists a basis $\lambda_1, \dots, \lambda_{2g}$ for Λ in terms of which E is given by the matrix*

$$E = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}, \quad \Delta = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_g \end{pmatrix},$$

where $d_i \in \mathbb{Z}$, $d_i > 0$ and $d_i | d_{i+1}$.

Definition 1.1.5. A basis of Λ with the property described in the Lemma is called symplectic basis of Λ for E .

Definition 1.1.6. Let $A = V/\Lambda$ be an Abelian variety and $E = c_1(L)$ a polarization on A . We say that E is a polarization of type $D = (d_1, \dots, d_g)$ if there exists a symplectic basis of Λ in terms of which E is in the form of the lemma and Δ is the diagonal matrix given by D .

We say that a (A, E) is a D -polarized Abelian variety if (A, E) is a polarized Abelian variety and E is of type D .

1.1.2 Period matrices

Let $T = V/\Lambda$ be a complex torus of dimension g and fix bases e_1, \dots, e_g for V and $\lambda_1, \dots, \lambda_{2g}$ for Λ . We can write:

$$\lambda_j = \sum_{i=1}^g \lambda_{i,j} e_i,$$

where $\lambda_{i,j} \in \mathbb{C}$. The matrix $\Pi \in \text{Mat}(g, 2g, \mathbb{C})$ given by

$$\Pi = \begin{pmatrix} \lambda_{1,1} & \dots & \lambda_{1,g} & \lambda_{1,g+1} & \dots & \lambda_{1,2g} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda_{g,1} & \dots & \lambda_{g,g} & \lambda_{g,g+1} & \dots & \lambda_{g,2g} \end{pmatrix},$$

is called a period matrix for T . By construction a period matrix contains by columns a basis of the lattice Λ in coordinates with respect to a basis of V . Notice that the period matrix encodes all the information about the complex torus T , but it depends on the choosen basis of V and Λ .

Proposition 1.1.7 (Riemann-Frobenius conditions). *A complex torus $T = V/\Lambda$ admits a polarization E of type D if and only if there exist basis e_1, \dots, e_g of V and $\lambda_1, \dots, \lambda_{2g}$ of Λ such that the period matrix Π has the form*

$$\Pi = (\tau, \Delta_D),$$

where Δ_D is the diagonal matrix given by D and $\tau \in \text{Mat}(g, \mathbb{C})$ is a symmetric matrix with $\text{Im}(\tau) > 0$.

In this basis E is given by the matrix

$$E_D = \begin{pmatrix} 0 & \Delta_D \\ -\Delta_D & 0 \end{pmatrix}.$$

Definition 1.1.8. The Siegel upper half space of degree g is the complex manifold:

$$\mathbb{H}_g := \{\tau \in \text{Mat}(g, \mathbb{C}) : {}^t\tau = \tau, \text{Im}(\tau) > 0\}.$$

Proposition 1.1.9 ([BL04, Proposition 8.1.2]). *Given a type D , the Siegel upper half space \mathbb{H}_g is a moduli space for polarized Abelian varieties of type D with symplectic basis.*

From now on, we will consider the transpose of the period matrix: in order to avoid misunderstanding we will denote the transpose matrix with Ω , reserving the letter Π for the classical period matrix.

1.1.3 Endomorphisms

Definition 1.1.10. Let $T = V/\Lambda$ and $T' = V'/\Lambda'$ be two complex tori. A morphism $f : T \rightarrow T'$ that send 0_T to $0_{T'}$ is called a homomorphism of complex tori. An endomorphism of a complex torus T is a homomorphism from T to itself.

Let $T = V/\Lambda$ be a complex torus. We denote by $\text{End}(T)$ the group of the endomorphisms of T . Since T is an Abelian group, $\text{End}(T)$ is in a natural way an associative unitary ring.

Let $f \in \text{End}(T)$. Then f determines two maps: $f_a : V \rightarrow V$ called the analytic representation of f , and $f_r : \Lambda \rightarrow \Lambda$ called the rational representation of f . Given basis for V and Λ , the map $f_a : V \rightarrow V$ determines a matrix $\rho_a(f) \in \text{Mat}(g, \mathbb{C})$ while the map $f_r : \Lambda \rightarrow \Lambda$ determines a matrix $\rho_r(f) \in \text{Mat}(2g, \mathbb{Z})$ such that

$$\rho_r(f)\Omega = \Omega\rho_a(f)$$

where Ω is the transpose of the period matrix. On the other hand given two matrices $M \in \text{Mat}(2g, \mathbb{Z})$ and $N \in \text{Mat}(g, \mathbb{C})$ such that $M\Omega = \Omega N$ there exists an $f \in \text{End}(T)$ with $\rho_r(f) = M$ and $\rho_a(f) = N$.

Remark 1.1.1. Since we consider the transpose of the period matrix the identification of f_a and f_r with the matrix $\rho_a(f)$ and $\rho_r(f)$ is given by the right (and not left) action, i.e. we get:

$$f_a(v) = v\rho_a(f), \quad f_r(\lambda) = \lambda\rho_r(f).$$

where $v \in V$ and $\lambda \in \Lambda$ are considered as row vectors.

1.1.4 Quaternion Algebras

Let $a, b \in \mathbb{Q}^*$. The central simple algebra

$$\frac{(a, b)}{\mathbb{Q}} = \mathbb{Q} \oplus \mathbf{i}\mathbb{Q} \oplus \mathbf{j}\mathbb{Q} \oplus \mathbf{k}\mathbb{Q}$$

with $\mathbf{i}^2 = a$, $\mathbf{j}^2 = b$ and $\mathbf{k} := \mathbf{ij} = -\mathbf{ji}$ is called quaternion algebra over \mathbb{Q} . A quaternion algebra is equipped with an anti-involution $x \mapsto \bar{x}$:

$$x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \mapsto \bar{x} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k}.$$

A quaternion algebra $(a, b)/\mathbb{Q}$ is either a skew field or it is isomorphic to the matrix algebra $\text{Mat}(2, \mathbb{Q})$.

Proposition 1.1.11 ([vG00, Section 7.2]). *The quaternion algebra $(a, b)/\mathbb{Q}$ is isomorphic to the matrix algebra $\text{Mat}(2, \mathbb{Q})$ if and only if the equation*

$$ax^2 + by^2 - abz^2 = 0$$

has a non-trivial solution $(x, y, z) \in \mathbb{Q}^3$.

Definition 1.1.12. A quaternion algebra B over \mathbb{Q} is called definite if $B_{\mathbb{R}} = B \otimes_{\mathbb{Q}} \mathbb{R}$ is a skew field; in particular it follows that $B_{\mathbb{R}}$ is the classical quaternion algebra constructed by Hamilton. A quaternion algebra B over \mathbb{Q} is called indefinite if $B_{\mathbb{R}} = B \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to $\text{Mat}(2, \mathbb{R})$.

Remark 1.1.2. Let $B = (a, b)/\mathbb{Q}$ be a quaternion algebra. If $ab < 0$ then B is totally indefinite.

1.2 Abelian surfaces with automorphism

1.2.1 Products of Elliptic Curves

The selfproduct of an elliptic curve with an automorphism of order three and four respectively provides, for any integer $d > 0$, a $(1, d)$ -polarized Abelian surface with an automorphism of the same order whose eigenvalue on $H^{2,0}$ is equal to one.

To see this, let $\zeta_j := e^{2\pi i/j}$ be a primitive j -th root of unity. For $j = 3, 4$, let E_j be the elliptic curve with an automorphism $f_j \in \text{End}(E_j)$ of order j :

$$E_j := \frac{\mathbb{C}}{\mathbb{Z} + \zeta_j\mathbb{Z}}, \quad f_j : E_j \longrightarrow E_j, \quad z \longmapsto \zeta_j z.$$

Then the Abelian surface $A_j := E_j^2$ has the automorphism

$$\phi_j := f_j \times f_j^{-1} \quad \phi_j : A_j := E_j \times E_j \longrightarrow A_j.$$

As f_j^* acts as multiplication by ζ_j on $H^{1,0}(E_j) = \mathbb{C}dz$, the eigenvalues of ϕ_j^* on $H^{1,0}(A_j)$ are ζ_j, ζ_j^{-1} . Thus ϕ_j^* acts as the identity on $H^{2,0}(A_j) \cong \wedge^2 H^{1,0}(A_j)$.

The principal polarization on E_j is fixed by f_j , so the product of this polarization on the first factor with d -times the principal polarization on the second factor is a $(1, d)$ -polarization on A_j which is invariant under ϕ_j .

The lattice $\Lambda_j \subset \mathbb{C}^2$ defining A_j is given by the image of the transpose period matrix Ω_j :

$$A_j \cong \frac{\mathbb{C}^2}{\Lambda_j}, \quad \Lambda_j = \mathbb{Z}^4 \Omega_j, \quad \Omega_j := \begin{pmatrix} \zeta_j & 0 \\ 0 & d\zeta_j \\ 1 & 0 \\ 0 & d \end{pmatrix}.$$

Consider the matrices $\rho_a(\phi_j)$ and $\rho_r(\phi_j)$, as defined in section 1.1.3. Here we have

$$\rho_r(\phi_j)\Omega_j = \Omega_j\rho_a(\phi_j), \quad \rho_r(\phi_j) = M_j, \quad \rho_a(\phi_j) = \begin{pmatrix} \zeta_j & 0 \\ 0 & \zeta_j^{-1} \end{pmatrix},$$

where the matrix M_j is given by:

$$M_3 := \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}, \quad M_4 := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

By the Riemann-Frobenius conditions (Proposition 1.1.7) the $(1, d)$ -polarization is defined by the alternating matrix E_d

$$E_d = \begin{pmatrix} 0 & \Delta_d \\ -\Delta_d & 0 \end{pmatrix}, \quad \Delta_d = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}.$$

The $(1, d)$ -polarization is preserved by ϕ_j since $\phi_j^* E_d = M_j E_d^t M_j = E_d$ (notice that by Remark 1.1.1 M_j acts on the lattice from the right).

1.2.2 Deformations

Definition 1.2.1. Given $M \in \text{Mat}(4, \mathbb{R})$ such that $M E_d^t M = E_d$ we define

$$M *_d \tau := (A\tau + B\Delta_d)(C\tau + D\Delta_d)^{-1} \Delta_d \quad \text{where } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Since $M_j E_d^t M_j = E_d$ for all $d \in \mathbb{N}$, we get the $*_d$ -action of M_j on \mathbb{H}_2 . We denote by $\mathbb{H}_{j,d}$ the fixed point set of M_j for this action:

$$\mathbb{H}_{j,d} := \{\tau \in \mathbb{H}_2 : M_j *_d \tau = \tau\}.$$

Proposition 1.2.2. *The $(1, d)$ -polarized Abelian surface $(A_{\tau, d}, E_d)$, $A_{\tau, d} = \mathbb{C}^2/(\mathbb{Z}^4\Omega_\tau)$ with $\tau \in \mathbb{H}_2$, admits an automorphism ϕ_j induced by M_j if and only if $\tau \in \mathbb{H}_{j, d}$. Moreover, $\mathbb{H}_{j, d}$ is biholomorphic to \mathbb{H}_1 , the Siegel space of degree one.*

Proof. The Abelian surface $A_{\tau, d} = \mathbb{C}^2/(\mathbb{Z}^4\Omega_\tau)$ admits an automorphism induced by M_j if and only if there exists a matrix $N_\tau \in \text{Mat}(2, \mathbb{C})$ such that

$$M_j\Omega_\tau = \Omega_\tau N_\tau.$$

Writing M_j and Ω_τ as block matrices, the equation is equivalent to the linear system

$$\begin{cases} A\tau + B\Delta_d = \tau N_\Omega, \\ C\tau + D\Delta_d = \Delta_d N_\tau. \end{cases}$$

Obtaining N_τ from the second equation and substituting in the first one we get:

$$(A\tau + B\Delta_d)(C\tau + D\Delta_d)^{-1}\Delta_d = \tau.$$

Conversely, if $M_j *_d \tau = \tau$ we define

$$N_\tau := \Delta_d^{-1}(C\tau + D\Delta_d)$$

Now it is immediate to prove that N_τ verify the linear system above, and then that $M_j\Omega_\tau = \Omega_\tau N_\tau$. The fact that this fixed point set is a copy of \mathbb{H}_1 in \mathbb{H}_2 follos easily from [Fre83, 5.12, p.196] \square

1.2.3 Polarizations and automorphisms

Let $A = V/\Lambda$ be a complex torus of dimension g . As in section 1.1.1 we identify V with $\Lambda_{\mathbb{R}}$ and we denote by $J : \Lambda_{\mathbb{R}} \rightarrow \Lambda_{\mathbb{R}}$ the almost complex structure induced by the multiplication by i on V . We fix basis e_1, \dots, e_g of V and $\lambda_1, \dots, \lambda_{2g}$ of Λ and we identify V with \mathbb{C}^g and Λ with \mathbb{Z}^{2g} .

An endomorphism of A corresponds to a \mathbb{C} -linear map M on \mathbb{C}^g such that $M\Lambda \subset \Lambda$:

$$\text{End}(A) \cong \{M \in \text{Mat}(2g, \mathbb{Z}) : JM = MJ\},$$

where $\text{Mat}(2g, \mathbb{Z})$ is the algebra of $2g \times 2g$ matrices with integer coefficients.

The Nèron Severi group of A , a subgroup of

$$H^2(A, \mathbb{Z}) \cong \wedge^2 H^1(A, \mathbb{Z}) \cong \wedge^2 \text{Hom}(\Lambda, \mathbb{Z}),$$

can be described similarly:

$$\text{NS}(A) \cong \{F \in \text{Mat}(2g, \mathbb{Z}) : {}^t F = -F, \quad JF {}^t J = F\},$$

where the alternating matrix $F \in \text{NS}(A)$ defines the bilinear form $(x, y) \mapsto xF {}^t y$. Moreover, F is a polarization, i.e. the first Chern class of an ample line bundle, if $F {}^t J$ is a positive definite matrix. In particular, F is then invertible (in $\text{Mat}(2g, \mathbb{Q})$).

Lemma 1.2.3. *Let $E, F \in \text{NS}(A)$ and let assume that E is invertible in $\text{Mat}(2g, \mathbb{Q})$. Then $FE^{-1} \in \text{End}(A)_{\mathbb{Q}}$*

Proof. See [BL04, Proposition 5.2.1a] for an intrinsic description of the result. Since $FE^{-1} \in \text{Mat}(2g, \mathbb{Q})$, we only have to check if it commutes with J :

$$\begin{aligned} JFE^{-1} &= {}^t({}^tF {}^tJ)E^{-1} = {}^t(-J^{-1}F)E^{-1} = \\ &= F({}^tJ)^{-1}E^{-1} = F(E {}^tJ)^{-1} = F(J^{-1}E)^{-1} = FE^{-1}J. \end{aligned}$$

□

In Theorem 1.2.4 we show that if $\tau \in \mathbb{H}_{j,d}$ then the endomorphism algebra of the corresponding Abelian surface $\text{End}(A_{\tau,d})_{\mathbb{Q}}$ contains a quaternion algebra (and not just the field $\mathbb{Q}(\zeta_j)!$).

This is of course well known (see for example [BL04, Exercise 4, Section 9.4]), but we can also determine this quaternion algebra explicitly. It allows us to find infinitely many families of $(1, d)$ -polarized Abelian surfaces whose generic member is simple and whose endomorphism ring is an (explicitly determined) order in a quaternion algebra. To find the endomorphisms, we study first the Néron Severi group. Notice that in the proof of the following Theorem we do not need to know the period matrices of the deformations explicitly.

Theorem 1.2.4. *Let $j \in \{3, 4\}$ and let $\tau \in \mathbb{H}_{j,d}$, so that the Abelian surface $A_{\tau,d}$ has an automorphism ϕ_j induced by M_j (see Proposition 1.2.2). Then the endomorphism algebra of $A_{\tau,d}$ also contains an element ψ_j with $\psi_j^2 = d$. Moreover, for a general $\tau \in \mathbb{H}_{j,d}$ one has*

$$\text{End}(A_{\tau,d}) = \mathbb{Z}[\phi_j, \psi_j], \quad \text{End}(A_{\tau,d})_{\mathbb{Q}} \cong \frac{(-j, d)}{\mathbb{Q}}.$$

Proof. The Néron-Severi group of an Abelian surface A can be described as

$$\text{NS}(A) \cong H^2(A, \mathbb{Z}) \cap H^{1,1}(A) \cong \{\omega \in H^2(A, \mathbb{Z}) : (\omega, \omega_A^{2,0}) = 0\},$$

where (\cdot, \cdot) denotes the \mathbb{C} -linear extension to $H^2(A, \mathbb{C})$ of the intersection form on $H^2(A, \mathbb{Z})$ and we fixed an holomorphic 2-form on A , so that $H^{2,0}(A) = \mathbb{C}\omega_A^{2,0}$. The intersection form is invariant under automorphisms of A , so $(\phi_j^*x, \phi_j^*y) = (x, y)$ for all $x, y \in H^2(A, \mathbb{Z})$, where $A = A_{\tau,d}$. Moreover, by construction of ϕ_j , we have that $\phi_j^*\omega_A^{2,0} = \omega_A^{2,0}$, so $\omega_A^{2,0} \in H^2(A, \mathbb{C})^{\phi_j^*}$, the subspace of ϕ_j^* -invariant classes.

Therefore any integral class which is orthogonal to the ϕ_j^* -invariant classes is in particular orthogonal to $\omega_A^{2,0}$ and thus must be in $\text{NS}(A)$:

$$\begin{aligned} \left(H^2(A, \mathbb{Z})^{\Phi_j^*}\right)^{\perp} &:= \{\omega \in H^2(A, \mathbb{Z}) : (\omega, \theta) = 0, \\ &\quad \forall \theta \in H^2(A, \mathbb{Z}) \text{ with } \phi_j^*\theta = \theta\} \subset \text{NS}(A). \end{aligned}$$

The eigenvalues of ϕ_j^* on $H^1(A, \mathbb{C}) = H^{1,0}(A) \oplus \overline{H^{1,0}(A)}$ are ζ_j and ζ_j^{-1} , both with multiplicity two. Thus the eigenvalues of ϕ^* on $H^2(A, \mathbb{C}) = \wedge^2 H^1(A, \mathbb{C})$ are ζ_j^2, ζ_j^{-2} , with multiplicity one, and 1 with multiplicity 4. In particular $(H^2(A, \mathbb{Z})^{\phi_j^*})^\perp$ is a free \mathbb{Z} -module of rank 2, it is the kernel in $H^2(A, \mathbb{Z})$ of $(\phi_3^*)^2 + \phi_3^* + 1$ in case $j = 3$ and of $\phi_4^* + 1$ in case $j = 4$. Identifying $H^2(A, \mathbb{Z})$ with the alternating bilinear \mathbb{Z} -valued maps on $\Lambda_j \cong \mathbb{Z}^4$, the action of ϕ^* is given by $M_j \cdot F := M_j F^t M_j$, where F is an alternating 4×4 matrix with integral coefficients. It is now easy to find a basis $E_{j,1}, E_{j,2}$ of the \mathbb{Z} -module $(H^2(A, \mathbb{Z})^{\phi_j^*})^\perp$:

$$\begin{aligned} E_{3,1} &= \begin{pmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ -2 & 0 & 1 & 0 \end{pmatrix}, & E_{3,2} &= \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}; \\ E_{4,1} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, & E_{4,2} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Since E_d defines a polarization on A , the matrices $E_d^{-1} E_{j,k}$, $k = 1, 2$, are the images under ρ_r of elements in $\text{End}(A)_\mathbb{Q}$ (cf. [BL04, Proposition 5.2.1a]). In this way we found that for any $\tau \in \mathbb{H}_{j,d}$, the Abelian surface $A = A_{\tau,d}$ has an endomorphism ψ_j defined by the matrix $\rho_r(\psi_j)$ below:

$$\rho_r(\psi_3) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ d & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & d & 0 \end{pmatrix}, \quad \rho_r(\psi_4) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & d & 0 \\ 0 & 1 & 0 & 0 \\ -d & 0 & 0 & 0 \end{pmatrix},$$

where $\rho_r(\psi_3) = \frac{d}{3}(E_{3,1} - E_{3,2})E_d^{-1}$ and $\rho_r(\psi_4) = dE_{4,1}E_d^{-1}$. We set $\eta := 1 + 2M_3$ and $\mu := 2M_4$; then it follows

$$\begin{cases} \rho_r(\psi_3)^2 = d, \\ \eta^2 = -3, \\ \eta \rho_r(\psi_3) = -\rho_r(\psi_3) \eta, \end{cases} \quad \begin{cases} \rho_r(\psi_4)^2 = d, \\ \mu^2 = -4, \\ \mu \rho_r(\psi_4) = -\rho_r(\psi_4) \mu. \end{cases}.$$

Therefore $(-j, d)/\mathbb{Q} \subset \text{End}(A)_\mathbb{Q}$. As $(-j, d)/\mathbb{Q}$ is an indefinite quaternion algebra (so of type II), for general $\tau \in \mathbb{H}_{j,d}$ the Abelian surface $A = A_{\tau,d}$ has $(-j, d)/\mathbb{Q} = \text{End}(A)_\mathbb{Q}$ by [BL04, Theorem 9.9.1]. Therefore if $\phi \in \text{End}(A)$, then $\rho_r(\phi)$ is both a matrix with integer coefficients and it is a linear combination of I , $M_j = \rho_r(\phi_j)$, $\rho_r(\psi_j)$ and $M_j \rho_r(\psi_j)$ with rational coefficients. It is then easy to check that $\text{End}(A)$ is as stated in the theorem. \square

Remark 1.2.1. The quaternion algebras $(-1, d)/\mathbb{Q}$ and $(-4, d)/\mathbb{Q}$ are isomorphic for all $d \in \mathbb{Q}^*$.

Using Magma [Magma], we found that for the following $d \leq 20$ the quaternion algebras $(-1, d)/\mathbb{Q}$ and $(-3, d)/\mathbb{Q}$ are skew fields:

d	discriminant $\frac{(-1, d)}{\mathbb{Q}}$		d	discriminant $\frac{(-3, d)}{\mathbb{Q}}$
3, 6, 15	6	,	2, 6, 8, 14, 18	6
7, 14	14		5, 15, 20	15
11	22		10	10
19	38		11	33
			17	51

Moreover, for $d \leq 20$, $\text{End}(A)$ is never a maximal order in $(-1, d)/\mathbb{Q}$, and it is a maximal order in $(-3, d)/\mathbb{Q}$ if and only if $d = 2, 5, 11, 17$.

In particular, for $\tau \in \mathbb{H}_{3,2}$ the Abelian surface $A_{\tau,2}$ has a $(1, 2)$ -polarization invariant by an automorphism of order three induced by M_3 and $\text{End}(A_{\tau,2}) = \mathcal{O}_6$, the maximal order in the quaternion algebra with discriminant 6, for general $\tau \in \mathbb{H}_{3,2}$.

1.2.4 An explicit example

We construct an explicit example of a simple Abelian surface with an automorphism, using Proposition 1.2.2 and Theorem 1.2.4. Let us consider the matrix:

$$\Omega = \begin{pmatrix} \tau \\ \Delta_3 \end{pmatrix} = \begin{pmatrix} \frac{2i\sqrt{3}}{3} & i \\ i & 2i\sqrt{3} \\ 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

We observe that:

$${}^t\tau = \tau, \quad \text{Im}\tau \geq 0, \quad M_4 *_3 \tau = \tau.$$

Then $\tau \in \mathbb{H}_{4,3}$ and the polarized Abelian surface $(A_{\tau,3} = \mathbb{C}^2/(\mathbb{Z}^4\Omega), E_3)$ has an automorphism of order 4 by Proposition 1.2.2. In order to prove that $A_{\tau,3}$ is a simple Abelian surface we will use the Rupert Criterion (see [BL04, Section 10.6] or [Mar98]). We compute minors of dimension 2 of the matrix Ω :

$$y_{12} = -3, y_{13} = -i, y_{14} = 2i\sqrt{3}, y_{23} = -2i\sqrt{3}, y_{24} = 3i, y_{34} = 3.$$

The complex numbers 3, $-i$ and $2i\sqrt{3}$ are linearly independent over \mathbb{Q} , so the map α has rank 3. We obtain

$$E \cap E(\alpha) : \begin{cases} x_{12} = x_{34} \\ x_{13} = 3x_{24} \\ x_{14} = x_{23} \\ x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0. \end{cases}$$

As conic in \mathbb{P}^2 it is defined by $C : X^2 - 3Y^2 + Z^2 = 0$.

Lemma 1.2.5. *The conic $C : X^2 - 3Y^2 + Z^2 = 0$, $C \subset \mathbb{P}^2$ has no rational points.*

Proof. The equation $X^2 - 3Y^2 + Z^2 = 0$ is homogenous of degree 2, so it admits solutions in \mathbb{Q}^3 if and only if it admits solutions in \mathbb{Z}^3 : in other words we have to prove that C has no integer points.

The equation modulo 3 becomes $X^2 + Y^2 \equiv 0 \pmod{3}$: then $X \equiv 0 \pmod{3}$ and $Z \equiv 0 \pmod{3}$. In particular we deduce that $X^2 + Z^2$ is divisible by an even power of 3, while $3Y^2$ is divisible by an odd power of 3. We conclude that the equation has only the trivial solution $X = Y = Z = 0$ and hence C has no integer points. \square

By the Lemma $Q \cap E(\alpha)$ has no rational points and then by the Rupert Criterion ([Mar98, Proposition 2.11]) $A_{\tau,3}$ is simple. We proved:

Lemma 1.2.6. *The Abelian surface $A_{\tau,3}$ is simple and it has an automorphism of order 4.*

By Proposition 1.2.2 the automorphism is induced by the matrix M_4 . Consider the matrix

$$N = \begin{pmatrix} \frac{2i\sqrt{3}}{3} & i \\ -\frac{i}{3} & -\frac{2i\sqrt{3}}{3} \end{pmatrix}.$$

N is the unique matrix in $Mat(2, \mathbb{C})$ that satisfies:

$$M_4 \Omega = \Omega N.$$

Now we consider new basis on the complex vector space and on the lattice. Let A and R be the change of basis matrices:

$$A := \begin{pmatrix} \frac{2\sqrt{3}+3}{6} & \frac{1}{2} \\ \frac{-2\sqrt{3}+3}{6} & \frac{-1}{2} \end{pmatrix}, \quad R := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then in the new basis the equation above becomes:

$$\underbrace{RM_4R^{-1}}_{\tilde{M}} \cdot \underbrace{R\Omega A^{-1}}_{\tilde{\Omega}} = \underbrace{R\Omega A^{-1}}_{\tilde{\Omega}} \cdot \underbrace{ANA^{-1}}_{\tilde{N}}.$$

By direct calculation we get:

$$\tilde{N} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \tilde{M} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\tilde{\Omega} = \begin{pmatrix} 1 & 1 \\ i & -i \\ -i(-2\sqrt{3}+3) & i(-2\sqrt{3}-3) \\ -2\sqrt{3}+3 & -2\sqrt{3}-3 \end{pmatrix}.$$

Normalizing, i.e. changing once more the basis of the complex vector space, we get the equivalent matrix

$$\Omega' = \begin{pmatrix} -i & -\frac{2\sqrt{3}}{3} \\ \frac{2\sqrt{3}}{3} & -i \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Remark 1.2.2. This proves that [BL04, Lemma 13.4.3] is false: there exists a simple Abelian surface with an automorphism of order 4.

Remark 1.2.3. Also [BL04, Theorem 13.4.2] is false, since is based on [BL04, Lemma 13.4.3].

Theorem 1.2.7. *Let A be a simple Abelian surface and $\varphi \in \text{Aut}(A)$ a non trivial automorphism of A (i.e. $\varphi \neq \pm 1_A$) of finite order. Then $\text{ord}(\varphi) \in \{3, 4, 5, 6, 10\}$.*

Proof. In [BL04, Chapter 13] it is already proved that a finite automorphism φ , $\varphi \neq \pm 1_A$, of a simple Abelian surface A must have order $\text{ord}(\varphi) \in \{3, 4, 5, 6, 10\}$. We observe that all this cases occur. Up to isomorphism there exists a unique simple Abelian surface of CM-type that admits an automorphism of order 5, and then also of order 10, composing by -1_A . As we observe above a general surface in $\mathbb{H}_{4,3}$ admits an automorphism φ of order 4 such that $\varphi^2 = -1_A$. Finally a general surface in $\mathbb{H}_{3,2}$ admits an automorphism of order 3, and then also of order 6 composing by -1_A . \square

1.3 The level moduli space

1.3.1 Moduli Space of $(1, d)$ -polarized abelian surfaces

The integral symplectic group with respect to E_d is defined as

$$\Gamma_d := \{ M \in GL(4, \mathbb{Z}) : ME_d^t M = E_d \}.$$

This group acts on the Siegel space as defined in Definition 1.2.1

$$\Gamma_d \times \mathbb{H}_2 \longrightarrow \mathbb{H}_2, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} *_d \tau := (A\tau + B\Delta_d)(C\tau + D\Delta_d)^{-1}\Delta_d.$$

Notice that for $d = 1$ one finds the standard action of the symplectic group on \mathbb{H}_2 . The quotient space (in general a singular quasi-projective 3-dimensional algebraic variety) is the moduli space \mathcal{A}_d of pairs (A, H) where A is an Abelian surface and H is a polarization of type $(1, d)$ ([HKW93, Theorem 1.10(i)]).

For the study of this moduli space, and of certain “level” covers of it, we use the standard action of $Sp(4, \mathbb{R})$ on \mathbb{H}_2 which is $*_1$. Let $R_d \in Mat(4, \mathbb{Z})$ be the matrix

$$R_d := \begin{pmatrix} I & 0 \\ 0 & \Delta_d \end{pmatrix},$$

where $I \in Mat(2, \mathbb{Z})$ is the identity matrix. For all $M \in \Gamma_d$ and for all $\tau \in \mathbb{H}_2$ we get

$$(R_d^{-1} M R_d) *_1 \tau = M *_d \tau.$$

We define $\Gamma_{1,d}^0 := R_d^{-1} \Gamma_d R_d$. $\Gamma_{1,d}^0$ is a subgroup of the (standard) real symplectic group $Sp(4, \mathbb{R})$. Therefore

$$\mathcal{A}_d := \Gamma_d \backslash \mathbb{H}_2 \cong \Gamma_{1,d}^0 \backslash \mathbb{H}_2,$$

where the actions are $*_d$ and $*_1$ respectively.

1.3.2 Congruence subgroups

We now follow [BL04] for the definition of coverings of the moduli space and maps to projective space. Recall that we defined a group Γ_d of matrices with integral coefficients which preserve the alternating form E_d . We will actually be interested in the form $2E_2$, which is preserved by the same group. With the notation from [BL04, Section 8.1, p.212] we thus have:

$$\Gamma_D := \Gamma_2 = Sp_4^D(\mathbb{Z}), \quad D = \text{diag}(2, 4) = 2\Delta_2.$$

It is easy to check that

$$\mathbb{Z}^4 \tilde{D}^{-1} = \{x \in \mathbb{Q}^4 : x(2E_2)y \in \mathbb{Z}, \forall y \in \mathbb{Z}^4\}, \quad \tilde{D} := \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}.$$

Let $T(2, 4)$ be the following quotient of \mathbb{Z}^4 :

$$T(2, 4) = (\mathbb{Z}^4 \tilde{D}^{-1}) / \mathbb{Z}^4 \cong (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})^2,$$

The group Γ_D acts on this quotient and we define

$$\Gamma_D(D) := \ker(\Gamma_D \longrightarrow \text{Aut}(T(2, 4))).$$

One verifies easily that

$$\begin{aligned} \Gamma_D(D) &= \{M \in \Gamma_D : \tilde{D}^{-1} M \equiv \tilde{D}^{-1} \text{ mod } Mat(4, \mathbb{Z})\} \\ &= \left\{ M = \begin{pmatrix} I + D\alpha & D\beta \\ D\gamma & I + D\delta \end{pmatrix} \in \Gamma_D : \alpha, \beta, \gamma, \delta \in Mat(2, \mathbb{Z}) \right\}. \end{aligned}$$

This shows that $\Gamma_D(D)$ is the subgroup as defined in [BL04, Section 8.3] (see also [BL04, Section 8.8]). The alternating form E_2 defines a “symplectic”

form $\langle \cdot, \cdot \rangle$ on $T(2, 4)$ with values in the fourth-roots of unity (cf. [Bar87, Section 3.1]). For this we write (cf. [Bar87, Section 2.1])

$$T(2, 4) = K \times \hat{K}, \quad K = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, \\ \hat{K} = \text{Hom}(K, \mathbb{C}^*) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z},$$

and the symplectic form is

$$\langle \cdot, \cdot \rangle: T(2, 4) \times T(2, 4) \longrightarrow \mathbb{C}^*, \quad \langle (\sigma, l), (\sigma', l') \rangle := l'(\sigma)l(\sigma')^{-1}.$$

We denote by $Sp(T(2, 4))$ the subgroup of $Aut(T(2, 4))$ of automorphisms which preserve this form.

Lemma 1.3.1. *The reduction homomorphism*

$$\Gamma_D \longrightarrow Sp(T(2, 4))$$

is surjective. Hence $\Gamma_D/\Gamma_D(D) \cong Sp(T(2, 4))$, this is a finite group of order $2^9 3^2$.

Proof. As the symplectic form is induced by E_2 , we have $im(\Gamma_D) \subset Sp(T(2, 4))$. In [Bar87, Proposition 3.1] generators ϕ_i , $i = 1, \dots, 5$ of $Sp(T(2, 4))$ are given. It is easy to check that the following matrices are in G_D and induce these automorphisms on $T(2, 4)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The order of $Sp(T(2, 4))$ is determined in [Bar87, Proposition 3.1]. □

1.3.3 The subgroup $\Gamma_D(D)_0$

We define a normal subgroup of $\Gamma_D(D)$ by:

$$\Gamma_D(D)_0 := \ker(\phi: \Gamma_D(D) \longrightarrow (\mathbb{Z}/2\mathbb{Z})^4), \\ \phi(M) = (\beta_0, \gamma_0) := (\beta_{11}, \beta_{22}, \gamma_{11}, \gamma_{22}),$$

where $M \in \Gamma_D(D)$ is as above. Since D has even coefficients, $D = 2diag(1, 2)$, it is easy to check that ϕ is a homomorphism. Moreover, ϕ is surjective since the matrix with $\alpha = \gamma = \delta = 0$ and $\beta = diag(a, b)$ ($a, b \in \mathbb{Z}$) is in $\Gamma_D(D)$ and maps to $(a, b, 0, 0)$, similarly the matrix with $\alpha = \beta = \delta = 0$ and $\gamma = diag(a, b)$ is also in $\Gamma_D(D)$ and maps to $(0, 0, a, b)$. It follows that $\Gamma_D(D)/\Gamma_D(D)_0 \cong (\mathbb{Z}/2\mathbb{Z})^4$.

The groups Γ_D , $\Gamma_D(D)$ and $\Gamma_D(D)_0$ are denoted by $G_{\mathbb{Z}}$, $G_{\mathbb{Z}}(e)$ and $G_{\mathbb{Z}}(e, 2e)$ in [Igu72, V.2, p.177]. In [Igu72, V.2 Lemma 4] one finds that $\Gamma_D(D)_0$ is in fact a normal subgroup of Γ_D . There is an exact sequence of groups:

$$0 \longrightarrow \Gamma_D(D)/\Gamma_D(D)_0 \longrightarrow \Gamma_D/\Gamma_D(D)_0 \longrightarrow \Gamma_D/\Gamma_D(D) \longrightarrow 0.$$

The group Γ_D acts on \mathbb{H}_2 in a natural way but to get the standard action $*_1$ one must conjugate these groups by a matrix R_D with diagonal blocks I , D and one obtains the groups

$$G_D = R_D^{-1} \Gamma_D R_D, \quad G_D(D) = R_D^{-1} \Gamma_D(D) R_D, \quad G_D(D)_0 = R_D^{-1} \Gamma_D(D)_0 R_D,$$

see [BL04, Section 8.8, 8.9].

The main result from [BL04, section 8.9] is Lemma 8.9.2 which asserts that the holomorphic map given by theta-null values

$$\psi_D : \mathbb{H}_2 \longrightarrow \mathbb{P}^7, \quad \tau \longrightarrow \left(\dots : \vartheta_{[0]}^l(0, \tau) : \dots \right)_{l \in K},$$

where l runs over $K = D^{-1}\mathbb{Z}^2/\mathbb{Z}^2$ and where the theta functions $\vartheta_{[0]}^l(v, \tau)$ are defined in [BL04, 8.5, Formula (1)], factors over a holomorphic map

$$\overline{\psi}_D : \mathcal{A}_D(D)_0 := \mathbb{H}_2/\Gamma_D(D)_0 \cong \mathbb{H}_2/G_D(D)_0 \longrightarrow \mathbb{P}^7.$$

1.3.4 Group actions

The finite group $\Gamma_D/\Gamma_D(D)_0$ acts on $\mathcal{A}_D(D)_0$. The Heisenberg group $\mathcal{H}(D)$, a non-Abelian central extension of $T(2, 4)$ by \mathbb{C}^* , acts on \mathbb{P}^7 ([BL04, Section 6.6]). This action is induced by irreducible representation (called the Schrödinger representation) of $\mathcal{H}(D)$ on the vector space $V(2, 4)$ of complex valued functions on the subgroup K of $T(2, 4)$ ([BL04, Section 6.7])

$$\rho_D : \mathcal{H}(D) \longrightarrow GL(V(2, 4)).$$

In [Bar87, Section 2.1]) the action of generators of $\mathcal{H}(D)$ on $\mathbb{P}V(2, 4) = \mathbb{P}^7$ are given explicitly, also the linear map $\tilde{\iota} \in GL(V(2, 4))$ which sends the delta functions $\delta_l \mapsto \delta_{-l}$ ($l \in K$) is introduced there (cf. Sections 1.4.1, 1.4.2).

The normalizer of the Heisenberg group (in the Schrödinger representation) is, by definition, the group

$$N(\mathcal{H}(D)) := \{ \gamma \in \text{Aut}(\mathbb{P}V(2, 4)) : \gamma \rho_D(\mathcal{H}(D)) \gamma^{-1} \subset \rho_D(\mathcal{H}(D)) \}.$$

The group $N(\mathcal{H}(D))$ maps onto $Sp(T(2, 4))$ with kernel isomorphic to $T(2, 4)$. The elements in this kernel are obtained as interior automorphisms: $\gamma = \rho_D(h)$, for some $h \in \mathcal{H}(D)$. Explicit generators of $N(\mathcal{H}(D))$ are given in [Bar87, Table 8] (but there seem to be some misprints in the action of the generators on $\mathcal{H}(D)$ in the lower left corner of that table). Let $N(\mathcal{H}(D))_2$ be the subgroup of $N(\mathcal{H}(D))$ of elements which commute with $\tilde{\iota}$. The group $N(\mathcal{H}(D))_2$ is an extension of $Sp(T(2, 4))$ by the 2-torsion subgroup (isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$) of $T(2, 4)$ and $\sharp N(\mathcal{H}(D))_2 = 2^{13}3^2$.

We need the following result.

Proposition 1.3.2. *There is an isomorphism $\gamma : G_D/G_D(D)_0 \cong N(\mathcal{H}(D))_2$, $M' \mapsto \gamma_{M'}$ such that the map $\bar{\psi}_D$ is equivariant for the action of these groups. So if we denote by $\tilde{\gamma}$ the composition*

$$\tilde{\gamma} : \Gamma_D/\Gamma_D(D)_0 \xrightarrow{\cong} G_D/G_D(D)_0 \xrightarrow{\gamma} N(\mathcal{H}(D)) ,$$

then $\bar{\psi}_D(M * \tau) = \tilde{\gamma}_M \bar{\psi}_D(\tau)$ where $*$ denotes the action of $\Gamma(D)$ on \mathbb{H}_2 .

Proof. Let $\mathcal{L}_\tau = L(H, \chi_0)$ be the line bundle on $A_{\tau,2} := \mathbb{C}^2/(\mathbb{Z}^4\Omega_\tau)$ which has Hermitian form H with $E_2 = \text{Im}H$ (so it defines a polarization of type $(1,2)$) and the quasi-character χ_0 is as in [BL04, 3.1, Formula (3)] for the decomposition $\Lambda = \mathbb{Z}^2\tau \oplus \mathbb{Z}^2\Delta_2$. According to [BL04, Remark 8.5.3d], the theta functions $\vartheta_{[0]}^l(v, \tau)$ are a basis of the vector space of classical theta functions for the line bundle $\mathcal{L}_\tau^{\otimes 2}$. As χ_0 takes values in $\{\pm 1\}$ one has $\mathcal{L}_\tau^{\otimes 2} = L(2H, \chi_0^2 = 1)$, so it is the unique line bundle with first Chern class $2E_2$ and trivial quasi-character. Thus if $M \in G_D$ and $\tau' = M *_1 \tau$ then $\phi_M^* \mathcal{L}_\tau^{\otimes 2} \cong \mathcal{L}_{\tau'}^{\otimes 2}$, where $\phi_M : A_{\tau',2} \rightarrow A_{\tau,2}$ is the isomorphism defined by M . Notice that \mathcal{L}_τ and $\mathcal{L}_\tau^{\otimes 2}$ are symmetric line bundles ([BL04, Corollary 2.3.7]).

Let $\mathcal{G}(\mathcal{L}_\tau^{\otimes 2})$ be the theta group ([BL04, Section 6.1]), it has an irreducible linear representation $\tilde{\rho}$ on $H^0(A_{\tau,2}, \mathcal{L}_\tau^{\otimes 2})$ ([BL04, Section 6.4]).

A theta structure $b : \mathcal{G}(\mathcal{L}_\tau^{\otimes 2}) \rightarrow \mathcal{H}(D)$ is an isomorphism of groups which is the identity on their subgroups \mathbb{C}^* . A theta structure b defines an isomorphism β_b , unique up to scalar multiple ([BL04, Section 6.7]), which intertwines the actions of $\mathcal{G}(\mathcal{L}_\tau^{\otimes 2})$ and $\mathcal{H}(D)$:

$$\beta_b : H^0(A_{\tau,2}, \mathcal{L}_\tau^{\otimes 2}) \longrightarrow V(2,4), \quad \beta_b \tilde{\rho}(g) = \rho_D(b(g)) \beta_b \quad (\forall g \in \mathcal{G}(\mathcal{L}_\tau^{\otimes 2})) .$$

A symmetric theta structure ([BL04, Section 6.9]) is a theta structure that is compatible with the action of $(-1) \in \text{End}(A_{\tau,2})$ on the symmetric line bundle $\mathcal{L}_\tau^{\otimes 2}$ and the map $\tilde{\iota} \in GL(V(2,4))$ defined in [Bar87, Section 2.1].

For $\tau \in \mathbb{H}_2$, define an isomorphism $\beta_\tau : H^0(A_{\tau,2}, \mathcal{L}_\tau^{\otimes 2}) \rightarrow V(2,4)$ by sending the basis vectors $\vartheta_{[0]}^l(v, \tau)$ to the delta functions δ_l for $l \in K$. From the explicit transformation formulas for the theta functions under translations by points in $A_{\tau,2}$ one finds that for $g \in \mathcal{G}(\mathcal{L}_\tau^{\otimes 2})$ the map $\beta_\tau \tilde{\rho}(g) \beta_\tau^{-1}$ acts as an element, which we denote by $b_\tau(g)$, of the Heisenberg group $\mathcal{H}(D)$ acting on $V(2,4)$. This map $b = b_\tau : \mathcal{G}(\mathcal{L}_\tau^{\otimes 2}) \rightarrow \mathcal{H}(D)$ is a theta structure and $\beta_\tau \tilde{\rho}(g) = \rho_D(b_\tau(g)) \beta_\tau$, moreover it is symmetric since $\theta_{[0]}^l(-v, \tau) = \theta_{[0]}^{-l}(v, \tau)$.

For $M \in G_D$ and $\tau' = M *_1 \tau$ we have an isomorphism $\beta_{\tau'}$ and the composition $\gamma_M := \beta_{\tau'} \phi_M^* \beta_\tau^{-1} \in GL(V(2,4))$, is an element of $N(\mathcal{H})$ since ϕ_M^* induces an isomorphism $\mathcal{G}(\mathcal{L}_\tau^{\otimes 2}) \rightarrow \mathcal{G}(\mathcal{L}_{\tau'}^{\otimes 2})$. In fact $\gamma_M \in N(\mathcal{H})_2$ since the theta structures $\beta_\tau, \beta_{\tau'}$ are symmetric and ϕ_M commutes with (-1) on the abelian varieties.

From [BL04, Proposition 6.9.4] it follows that the group generated by the γ_M is contained in an extension of $Sp(T(2,4))$ by $(\mathbb{Z}/2\mathbb{Z})^4$. The map

$M \mapsto \gamma_M \in \text{Aut}(\mathbb{P}(V(2, 4)))$ is thus a (projective) representation of G_D whose image is contained in $N(\mathcal{H})_2$ and which, by construction, is equivariant for ψ_D . Unwinding the various definitions, we have shown that γ_M maps the point $(\dots : \theta_0^{[l]}(v, \tau) : \dots)$ to the point $(\dots : \theta_0^{[l]}(C\tau + D)v, M *_1 \tau) : \dots)$ where M has block form A, \dots, D . From the classical theory of transformations of theta functions (as in [BL04, Section 8.6]) one now deduces that $M \mapsto \gamma_M$ provides the desired isomorphism of groups. Notice that the element $-I \in G_D$, which acts trivially on \mathbb{H}_2 , maps to $\tilde{l} \in N(\mathcal{H}(D))_2$ which acts trivially on the subspace $\mathbb{P}^5 \subset \mathbb{P}^7$ of even theta functions. \square

1.4 A projective model of a Shimura curve

1.4.1 Barth's Variety $M_{2,4}$

We choose projective coordinates x_1, \dots, x_8 on $\mathbb{P}^7 = \mathbb{P}V(2, 4)$ as in [Bar87, Section 2.1]. The map $\tilde{l} \in \text{Aut}(\mathbb{P}^7)$ is then given by

$$\tilde{l}(x) = (x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : -x_7 : -x_8) .$$

It has two eigenspaces which correspond to the even and odd theta functions. The image of $\overline{\psi}_D$ lies in the subspace $\mathbb{P}^5 = \mathbb{P}V(2, 4)_+$ of even functions which is defined by $x_7 = x_8 = 0$. We use x_1, \dots, x_6 as coordinates on this \mathbb{P}^5 . Let

$$f_1 := -x_1^2 x_2^2 + x_3^2 x_4^2 + x_5^2 x_6^2, \quad f_2 := -(x_1^4 + x_2^4) + x_3^4 + x_4^4 + x_5^4 + x_6^4 .$$

Then Barth's variety of theta-null values is defined as ([Bar87, (3.9)])

$$M_{2,4} := \{x \in \mathbb{P}^5 : f_1(x) = f_2(x) = 0\} .$$

The image of $\overline{\psi}_D(\mathbb{H}_2)$ is a quasi-projective variety and the closure of its image is $M_{2,4}$.

1.4.2 The Heisenberg group action

Recall that $T(2, 4) = \mathbb{Z}^4 \tilde{D}^{-1} / \mathbb{Z}^4$ and let $\sigma_1, \sigma_2, \tau_1, \tau_2 \in T(2, 4)$ be the images of $e_1/2, e_2/4, e_3/2, e_4/4$. We denote certain lifts of the generators σ_1, \dots, τ_2 of $T(2, 4)$ to $\mathcal{H}(D)$ by $\tilde{\sigma}_1, \dots, \tilde{\tau}_2$. These lifts act, in the Schrödinger representation, on $\mathbb{P}^7 = \mathbb{P}V(2, 4)$ as follows (see [Bar87, Table 1]):

$$\begin{aligned} \tilde{\sigma}_1(x) &= (x_2 : x_1 : x_4 : x_3 : x_6 : x_5 : x_8 : x_7) , \\ \tilde{\sigma}_2(x) &= (x_3 : x_4 : x_1 : x_2 : x_7 : x_8 : -x_5 : -x_6) , \\ \tilde{\tau}_1(x) &= (x_1 : -x_2 : x_3 : -x_4 : x_5 : -x_6 : x_7 : -x_8) , \\ \tilde{\tau}_2(x) &= (x_5 : x_6 : ix_7 : ix_8 : x_1 : x_2 : ix_3 : ix_4) , \end{aligned}$$

where $x = (x_1 : \dots : x_8) \in \mathbb{P}^7$ and $i^2 = -1$. For any $g = (a, b, c, d) \in T(2, 4)$ one then finds the action of a lift \tilde{g} of g by defining $\tilde{g} := \tilde{\sigma}_1^a \cdots \tilde{\tau}_2^d$.

Proposition 1.4.1. *Let $\tilde{\mu}_3$ on \mathbb{P}^7 be the projective transformation defined as*

$$\begin{aligned} \tilde{\mu}_3 : x \mapsto & (x_3 - ix_4 : x_3 + ix_4 : \zeta x_5 - \zeta^3 x_6 : \zeta x_5 + \zeta^3 x_6 : \\ & x_1 - ix_2 : x_1 + ix_2 : \zeta^3 x_7 + \zeta x_8 : \zeta^3 x_7 - \zeta x_8), \end{aligned}$$

where ζ is a primitive 8-th root of unity (so $\zeta^4 = -1$) and $i := \zeta^2$. Then $\tilde{\mu}_3 \in N(\mathcal{H}(D))_2$ and with M_3 as in Section 1.2.1 we have

$$\tilde{\gamma}_{M_3} = \tilde{h}\tilde{\mu}_3\tilde{h}^{-1}$$

for some $\tilde{h} \in \ker(N(\mathcal{H}(D))_2 \rightarrow Sp(T(2, 4)))$.

Proof. The map $M_3 : \mathbb{Z}^4 \rightarrow \mathbb{Z}^4$ from section 1.2.1 induces the (symplectic) automorphism \overline{M}_3 of $T(2, 4)$ given by (recall that we used row vectors, so for example $e_4 M_3 = -e_2 - e_4$ and thus $\tau_2 \mapsto -\sigma_2 - \tau_2$):

$$\sigma_1 \longmapsto -\sigma_1 - \tau_1, \quad \sigma_2 \longmapsto \tau_2, \quad \tau_1 \longmapsto \sigma_1, \quad \tau_2 \longmapsto -\sigma_2 - \tau_2.$$

Now one verifies that, as maps on \mathbb{C}^8 , one has

$$\tilde{\mu}_3 \tilde{\sigma}_1 \tilde{\mu}_3^{-1} = i \tilde{\sigma}_1^{-1} \tilde{\tau}_1^{-1}, \quad \tilde{\mu}_3 \tilde{\sigma}_2 \tilde{\mu}_3^{-1} = \tilde{\tau}_2, \quad \tilde{\mu}_3 \tilde{\tau}_1 \tilde{\mu}_3^{-1} = \tilde{\sigma}_1, \quad \tilde{\mu}_3 \tilde{\tau}_2 \tilde{\mu}_3^{-1} = \zeta \tilde{\sigma}_2^{-1} \tilde{\tau}_2^{-1}.$$

Hence $\tilde{\mu}_3 \in Aut(\mathbb{P}^7)$ is in the normalizer $N(\mathcal{H})$ and it is a lift of $\overline{M}_3 \in Sp(T(2, 4))$. One easily verifies that it commutes with the action of $\tilde{\iota}$ on \mathbb{P}^7 so $\tilde{\mu}_3 \in N(\mathcal{H})_2$. Any other lift of \overline{M}_3 to $Aut(\mathbb{P}^7)$ which commutes with $\tilde{\iota}$ is of the form $\tilde{g}\tilde{\mu}_3$ for some $g \in T(2, 4)$ with $2g = 0$. Since $\overline{M}_3^2 + \overline{M}_3 + I = 0$, the map $h \mapsto (\overline{M}_3 + I)h$ is an isomorphism on the two-torsion points in $T(2, 4)$. Thus there is an $h \in T(2, 4)$, with $2h = 0$, such that $g = (\overline{M}_3 + I)h$. As $\tilde{\mu}_3 \tilde{h} \tilde{\mu}_3^{-1} = \tilde{k}$, where $k = \overline{M}_3 h$ and thus $k = g + h$, it follows that $\tilde{h}\tilde{\mu}_3\tilde{h}^{-1} = \tilde{g}\tilde{\mu}_3$. \square

1.4.3 Fixed Points and Eigenspaces

The map $\overline{\psi}_D$ is equivariant for the actions of Γ_D and $N(\mathcal{H})_2$. Hence the fixed points of M_3 in \mathbb{H}_2 , which parametrize abelian surfaces with quaternionic multiplication, map to the fixed points of $\tilde{\gamma}_{M_3} = \tilde{h}\tilde{\mu}_3\tilde{h}^{-1}$ in \mathbb{P}^7 . Conjugating M_3 by an element $N \in \Gamma_D$ such that $\tilde{\gamma}_N = \tilde{h}$ (as in Proposition 1.4.1), we obtain an element of order three $M'_3 \in \Gamma_D$ whose fixed point locus $\mathbb{H}_2^{M'_3}$ also consists of period matrices of Abelian surfaces with QM by \mathcal{O}_6 and the image $\overline{\psi}_D(\mathbb{H}_2^{M'_3})$ consists of fixed points of $\tilde{\mu}_3$. The following lemma identifies this fixed point set.

Theorem 1.4.2. *Let $\mathbb{P}_{QM}^1 \subset \mathbb{P}^5$ be the projective line parametrized by*

$$\mathbb{P}^1 \xrightarrow{\cong} \mathbb{P}_{QM}^1, \quad (x : y) \longmapsto p_{(x:y)} := (\sqrt{2}x : \sqrt{2}y : x+y : i(x-y) : x-iy : x+iy).$$

Then $\mathbb{P}_{QM}^1 \subset M_{2,4}$ is a Shimura curve that parametrizes Abelian surfaces with QM by \mathcal{O}_6 , the maximal order in the quaternion algebra of discriminant 6.

The following two elements $\tilde{\nu}_1, \tilde{\nu}_2 \in N(\mathcal{H}(D))_2$,

$$\begin{aligned}\tilde{\nu}_1(x) &= (x_5 + x_6, -x_5 + x_6, \zeta(x_3 - x_4), \zeta(x_3 + x_4), \\ &\quad x_1 + x_2, x_1 - x_2, \zeta(-x_7 + x_8), \zeta(x_7 + x_8)) \\ \tilde{\nu}_2(x) &= (x_4, -x_3, \zeta^3 x_6, \zeta^3 x_5, ix_1, -ix_2, \zeta^3 x_7, \zeta^3 x_8),\end{aligned}$$

restrict to maps in $\text{Aut}(\mathbb{P}_{QM}^1)$ which generate a subgroup isomorphic to the symmetric group $\mathcal{S}_4 \subset \text{Aut}(\mathbb{P}_{QM}^1)$.

Proof. The subspace \mathbb{P}^5 is mapped into itself by $\tilde{\mu}_3$. The restriction μ_3 of $\tilde{\mu}_3$ to \mathbb{P}^5 has three eigenspaces on \mathbb{C}^6 , each 2-dimensional. The eigenspace of μ_3 with eigenvalue $\sqrt{2} := \zeta + \zeta^7$ is the only eigenspace whose projectivization \mathbb{P}_{QM}^1 is contained in $M_{2,4}$. Thus $\overline{\psi}_D(\mathbb{H}_2^{M_3}) \subset \mathbb{P}_{QM}^1$ and we have equality since the locus of Abelian surfaces with QM by \mathcal{O}_6 in $\mathcal{A}_D(D)_0$ (in fact in any level moduli space) is known to be a compact Riemann surface.

The maps $\tilde{\nu}_1, \tilde{\nu}_2$ commute with $\tilde{\iota}$ and moreover:

$$\begin{aligned}\tilde{\nu}_1 \tilde{\sigma}_1 \tilde{\nu}_1^{-1} &= -\tilde{\sigma}_1 \tilde{\tau}_2^2, & \tilde{\nu}_1 \tilde{\sigma}_2 \tilde{\nu}_1^{-1} &= i \tilde{\sigma}_1 \tilde{\sigma}_2^2 \tilde{\tau}_2, \\ \tilde{\nu}_1 \tilde{\tau}_1 \tilde{\nu}_1^{-1} &= -\tilde{\sigma}_2^2 \tilde{\tau}_1, & \tilde{\nu}_1 \tilde{\tau}_2 \tilde{\nu}_1^{-1} &= \zeta \tilde{\sigma}_1 \tilde{\sigma}_2^3 \tilde{\tau}_1 \tilde{\tau}_2, \\ \tilde{\nu}_2 \tilde{\sigma}_1 \tilde{\nu}_2^{-1} &= -\tilde{\tau}_1 \tilde{\tau}_2^2, & \tilde{\nu}_2 \tilde{\sigma}_2 \tilde{\nu}_2^{-1} &= \zeta \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\tau}_2, \\ \tilde{\nu}_2 \tilde{\tau}_1 \tilde{\nu}_2^{-1} &= -\tilde{\sigma}_1 \tilde{\sigma}_2^2 \tilde{\tau}_2^2, & \tilde{\nu}_2 \tilde{\tau}_2 \tilde{\nu}_2^{-1} &= \tilde{\tau}_1 \tilde{\tau}_2^3,\end{aligned}$$

hence they are in $N(\mathcal{H})_2$. The maps ν_1, ν_2 have order 4 and 3 respectively in $\text{Aut}(\mathbb{P}^7)$ and map \mathbb{P}_{QM}^1 into itself. In fact, the induced action on \mathbb{P}_{QM}^1 is $\tilde{\nu}_i p(x:y) = p_{\nu_i(x:y)}$ with

$$\nu_1(x : y) := (x : iy), \quad \nu_2(x : y) := (i(x - y) : -(x + y)) .$$

We verified that $\nu_1, \nu_2 \in \text{Aut}(\mathbb{P}^1)$ generate a subgroup which is isomorphic to the symmetric group \mathcal{S}_4 (to obtain this isomorphism, one may use the action of the ν_i on the four irreducible factors in $\mathbb{Q}(\zeta)[x, y]$ of the polynomial g_8 defined in Corollary 1.4.2.1). \square

Corollary 1.4.2.1. *The images in \mathbb{P}_{QM}^1 under the parametrization given in Proposition 1.4.2 of the zeroes of the polynomials*

$$\begin{aligned}g_6 &:= xy(x^4 - y^4), & g_8 &:= x^8 + 14x^4y^4 + y^8, \\ g_{12} &:= x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12},\end{aligned}$$

are the orbits of the points in \mathbb{P}_{QM}^1 with a non-trivial stabilizer in \mathcal{S}_4 . Moreover, the rational function

$$G := g_6^4/g_8^3 : \mathbb{P}_{QM}^1 \longrightarrow \mathbb{P}^1 \cong \mathbb{P}_{QM}^1/\mathcal{S}_4$$

defines the quotient map by \mathcal{S}_4 .

Proof. A non-trivial element σ in $\mathcal{S}_4 \subset \text{Aut}(\mathbb{P}_{QM}^1)$ has two fixed points, corresponding to the eigenlines of any lift of σ to $GL(2, \mathbb{C})$. The fixed points of σ^k are the same as those of σ whenever σ^k is not the identity on \mathbb{P}_{QM}^1 . One now easily verifies that the fixed points of cycles of order 3, 4, 2 are the zeroes of g_6, g_8, g_{12} respectively.

The quotient map $\mathbb{P}_{QM}^1 \rightarrow \mathbb{P}_{QM}^1/\mathcal{S}_4 \cong \mathbb{P}^1$ has degree 24. The rational function $G := g_6^4/g_8^3$ is \mathcal{S}_4 -invariant and defines a map of degree 24 from \mathbb{P}_{QM}^1 to \mathbb{P}^1 , hence the quotient map is given by G . \square

1.5 The Principal Polarization

1.5.1 Introduction

In the previous section we considered Abelian surfaces whose endomorphism ring contains \mathcal{O}_6 endowed with a $(1, 2)$ -polarization. Rotger proved that an Abelian surface A whose endomorphism ring is \mathcal{O}_6 admits a principal polarization; moreover this polarization is essentially unique, i.e. there exists a unique curve C of genus 2 such that $A = \text{Jac}(C)$ (see [Rot04, section 7]). The Abel-Jacobi image of the genus two curve provides the principal polarization. In this section we find the image of such a curve in the Kummer surface. This allows us to relate these genus two curves to the ones described by Hashimoto and Murabayashi in [HM95] in Section 1.5.6.

Moreover, we also find an explicit projective model of a surface in the moduli space $M_{2,4}$ which parametrizes $(2, 4)$ -polarized Abelian surfaces whose endomorphism ring contains $\mathbb{Z}[\sqrt{2}]$, see Section 1.5.8.

1.5.2 Polarizations

To explain how we found genus two curves in the $(2, 4)$ -polarized Abelian surfaces parametrized by \mathbb{P}_{QM}^1 , it is convenient to first consider the Jacobian $A = \text{Pic}^0(C)$ of one of the genus two curves given in [HM95, Theorem 1.3]. In [HM95, Section 3.1] and in [Dol14, Lecture 7] one finds an explicit description of the principal polarization E .

Consider the indefinite quaternion algebra B_6

$$B_6 = \frac{(-6, 2)}{\mathbb{Q}},$$

and notice that

$$\frac{(-6, 2)}{\mathbb{Q}} \simeq \frac{(-3, 2)}{\mathbb{Q}}.$$

As explained in Section 1.1.4 B_6 is generated, as \mathbb{Q} -algebra, by two elements \mathbf{i} and \mathbf{j} such that

$$\mathbf{i}^2 = -6, \quad \mathbf{j}^2 = 2, \quad \mathbf{ij} = -\mathbf{ji}.$$

As \mathbb{Q} vector space it is generated by $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$, where we set $\mathbf{k} := \mathbf{ij}$. Consider the maximal order \mathcal{O}_6 given by:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(1, \frac{\mathbf{i} + \mathbf{j}}{2}, \frac{\mathbf{i} - \mathbf{j}}{2}, \frac{2 + 2\mathbf{j} + \mathbf{k}}{4}\right).$$

Remark 1.5.1. Via an appropriate isomorphism $(-6, 2)/\mathbb{Q} \cong (-3, 2)/\mathbb{Q}$ this is the same maximal order found in Section 1.2.3.

We define an embedding of $B_{\mathbb{R}} = B \otimes_{\mathbb{Q}} \mathbb{R}$ into $Mat(2, \mathbb{R})$ by

$$\mathbf{i} \mapsto \begin{pmatrix} 0 & -1 \\ 6 & 0 \end{pmatrix}, \quad \mathbf{j} \mapsto \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix},$$

and we consider the linear map

$$\Phi : B_{\mathbb{R}} \rightarrow \mathbb{C}^2, \quad X \mapsto X \cdot \underline{z},$$

where $\underline{z} = {}^t(z, 1)$ with $z \in \mathbb{C} \setminus \mathbb{R}$. We obtain:

$$\begin{aligned} \omega_1 &:= \Phi(\alpha_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} z \\ 1 \end{pmatrix}, \\ \omega_2 &:= \Phi(\alpha_2) = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{1}{2} \\ 3 & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2}z - \frac{1}{2} \\ 3z - \frac{\sqrt{2}}{2} \end{pmatrix}, \\ \omega_3 &:= \Phi(\alpha_3) = \begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{1}{2} \\ 3 & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2}z - \frac{1}{2} \\ 3z + \frac{\sqrt{2}}{2} \end{pmatrix}, \\ \omega_4 &:= \Phi(\alpha_4) = \begin{pmatrix} \frac{1+\sqrt{2}}{2} & \frac{\sqrt{2}}{4} \\ \frac{3\sqrt{2}}{2} & \frac{1-\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{2}}{2}z + \frac{\sqrt{2}}{4} \\ \frac{3\sqrt{2}}{2}z + \frac{1-\sqrt{2}}{2} \end{pmatrix}. \end{aligned}$$

We set $\Lambda_z := \Phi(\mathcal{O}_6) = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z} \oplus \omega_3\mathbb{Z} \oplus \omega_4\mathbb{Z}$. As observed in [HM95], Λ_z is a lattice in \mathbb{C}^2 and then we can construct the complex torus

$$A_z = \frac{\mathbb{C}^2}{\Lambda_z}.$$

We construct a bilinear form $E_z : \Lambda_z \times \Lambda_z \rightarrow \mathbb{Z}$ by

$$E_z(\omega_i, \omega_j) = Tr(-\mathbf{i}/6 \cdot \alpha_i \cdot \overline{\alpha_j}). \quad (1.1)$$

The form E_z defines a polarization on A_z and in the basis $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ of Λ_z is defined by the matrix $E \in Mat(4, \mathbb{Z})$, $\{E\}_{i,j} = E_z(\omega_i, \omega_j)$:

$$E = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Consider the following change of basis of Λ_z :

$$\begin{cases} \omega'_1 = -\omega_3, \\ \omega'_2 = \omega_4, \\ \omega'_3 = -\omega_1, \\ \omega'_4 = \omega_3 - \omega_2. \end{cases}$$

In the new basis $\{\omega'_1, \omega'_2, \omega'_3, \omega'_4\}$ E_z is represented by the matrix $\tilde{E} \in \text{Mat}(4, \mathbb{Z})$:

$$\tilde{E} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Then (A_z, E_z) is a principal polarized abelian surface.

By the Riemann-Frobenius conditions (Proposition 1.1.7) there exists a basis of Λ_z such that the period matrix of A_z is (τ_z, Id) , where

$${}^t\tau_z = \tau_z, \quad \text{Im}(\tau_z) > 0.$$

The basis $\{\omega'_1, \omega'_2, \omega'_3, \omega'_4\}$ is a symplectic basis of Λ_z for E_z ; then we simply choose ω'_3, ω'_4 as basis of \mathbb{C}^2 .

$$\begin{cases} \omega'_1 = -\omega_3 = a(-\omega_1) + b(\omega_3 - \omega_2), \\ \omega'_2 = \omega_4 = c(-\omega_1) + d(\omega_3 - \omega_2). \end{cases}$$

Solving the system above, we obtain:

$$\tau_z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{3z}{2} - \frac{1}{4z} & -\frac{3\sqrt{2}z}{4} - \frac{1}{2} - \frac{\sqrt{2}}{8z} \\ -\frac{3\sqrt{2}z}{4} - \frac{1}{2} - \frac{\sqrt{2}}{8z} & \frac{3z}{4} - \frac{1}{2} - \frac{1}{8z} \end{pmatrix}.$$

By construction we get $\text{End}(A_z) \supseteq \mathcal{O}_6$ and $\text{End}(A_z)_{\mathbb{Q}} \supseteq B_6$.

Consider the following elements in \mathcal{O}_6 :

$$\mu := \alpha_2 = (\mathbf{i} + \mathbf{j})/2, \quad \eta := (-2 + 2\mathbf{i} + \mathbf{k})/4. \quad (1.2)$$

The element μ has order 4, while η has order 3: this means that the Abelian surface A_z has automorphisms of order 3 and 4.

Remark 1.5.2. Since we are interested in the $(1, 2)$ -polarization we now focus on the automorphism of order three η . A similar construction can be repeated for the automorphism of order four μ .

With an abuse of notation we denote by η also the endomorphism defined by the element. Since E is a principal polarization also η^*E is and we obtain a polarization E' that is invariant under η as follows:

$$E' = E + \eta^*E + (\eta^2)^*E.$$

An explicit computation shows that E' is a polarization of type $(3, 6)$. We denote by E'' the $(1, 2)$ polarization such that $3E'' = E'$. Considering E as a class in $H^2(A, \mathbb{Z})$, one has $E^2 = 2$, since E is a principal polarization. As η is an automorphism of A , we also get

$$\begin{aligned} (\eta^*E)^2 &= ((\eta^2)^*E)^2 = 2, \\ E \cdot \eta^*E &= \eta^*E \cdot (\eta^2)^*E = (\eta^2)^*E \cdot E. \end{aligned}$$

Then one finds that $(E')^2 = 6 + 6(E \cdot \eta^*E)$ and as E' defines a polarization of type $(3, 6)$ we have $(E')^2 = 36$. Hence we get $E \cdot (\eta^*E) = 5$. In conclusion one finds that

$$E \cdot E'' = E \cdot (E + \eta^*E + (\eta^2)^*E)/3 = (2 + 5 + 5)/3 = 4.$$

Identify the Jacobian of the genus two curve C with $\text{Pic}^0(C) = A$ and identify C with its image under the Abel-Jacobi map $C \rightarrow \text{Pic}^0(C)$, $p \mapsto p - p_0$, where p_0 is a Weierstrass point. If the hyperelliptic involution interchanges the points $q, q' \in C$, then $q + q'$ and $2p_0$ are linearly equivalent and thus $q - p_0 = -(q' - p_0)$. Hence the curve $C \subset \text{Pic}^0(C)$ is symmetric: $(-1)^*C = C$. If p_1, \dots, p_5 are the other Weierstrass points of C , then $2p_i$ is linearly equivalent to $2p_0$, hence the five points $p_i - p_0 \in C \subset A$, $i = 1, \dots, 5$ are points of order two in A .

Let now \mathcal{L} be a symmetric line bundle on A defining the $(1, 2)$ -polarization E'' on A . As $E \cdot E'' = 4$, the restriction of \mathcal{L} to C has degree 4 and thus $\mathcal{L}^{\otimes 2}$ restricts to a degree 8 line bundle on C . The map given by the even sections $H^0(A, \mathcal{L}^{\otimes 2})_+$ defines a 2:1 map from A onto the Kummer surface $A/\pm 1$ of A in \mathbb{P}^5 . As $(2E'')^2 = 16$, this Kummer surface has degree $16/2 = 8$. In fact, Barth shows that the Kummer surface is the complete intersection of three quadrics, see Section 1.5.3. The symmetry of C implies that this image is a rational curve and the degree of the image of C is four. But a rational curve of degree four in a projective space spans at most a \mathbb{P}^4 . Moreover, this \mathbb{P}^4 contains at least six of the nodes (the images of the two-torsion points of A) of the Kummer surface which lie on C .

It should be noticed that any $(2, 4)$ -polarized Kummer surface in \mathbb{P}^5 contains subsets of four nodes which span only a \mathbb{P}^2 (cf. [GS13, Lemma 5.3]), these subsets must be avoided to find C .

Conversely, given a rational quartic curve on the Kummer surface which passes through exactly 6 nodes, its inverse image in the Abelian surface will be a genus two curve C . In fact, the general A is simple, hence there are no non-constant maps from a curve of genus at most one to A . The adjunction formula on A shows that $C^2 = 2$, hence C defines a principal polarization on A . Rotger [Rot04, Section 6] proved that an Abelian surface A with $\text{End}(A) = \mathcal{O}_6$ has a unique principal polarization up to isomorphism. Thus C must be a member of the family of genus two curves in given in [HM95,

Theorem 1.3]. We summarize the results in this section in the following proposition. In Proposition 1.5.2 we determine the curve from [HM95] which is isomorphic to $C = C_x$ on the Abelian surface defined by $x \in \mathbb{P}_{QM}^1$.

Proposition 1.5.1. *Let A be an Abelian surface with $\mathcal{O}_6 \subset \text{End}(A)$. Then A has a (unique up to isomorphism) principal polarization defined by a genus 2 curve $C \subset A$ which is isomorphic to a curve from the family in [HM95, Theorem 1.3] (see Section 1.5.6).*

There is an automorphism of order three $\eta \in \text{Aut}(A)$ such that

$$C + \eta^*C + (\eta^2)^*C = 3E''$$

defines a polarization of type $(3, 6)$. Let \mathcal{L} be a symmetric line bundle with $c_1(\mathcal{L}) = E''$. Then the image of C , symmetrically embedded in A , under the map $A \rightarrow \mathbb{P}^5$ defined by the subspace $H^0(A, \mathcal{L}^{\otimes 2})_+$, is a rational curve of degree four which passes through exactly six nodes of the Kummer surface of A which lie in a hyperplane in \mathbb{P}^5 .

Conversely, the inverse image in A of a rational curve which passes through exactly six nodes of the Kummer surface of A is a genus two curve which defines a principal polarization on A .

1.5.3 A reducible hyperplane section

Now we give a hyperplane $H_x \subset \mathbb{P}^5$ which cuts the Kummer surface K_x for $x \in \mathbb{P}_{QM}^1$ in two rational curves of degree four, the curves intersect in six points which are nodes of K_x .

A general point $x = (x_1 : \dots : x_6) \in M_{2,4} \subset \mathbb{P}^5$ defines a $(2, 4)$ -polarized Kummer surface K_x which is the complete intersection of the following three quadrics in X_1, \dots, X_6 :

$$\begin{aligned} q_1 &:= (x_1^2 + x_2^2)(X_1^2 + X_2^2) - (x_3^2 + x_4^2)(X_3^2 + X_4^2) - (x_5^2 + x_6^2)(X_5^2 + X_6^2), \\ q_2 &:= (x_1^2 - x_2^2)(X_1^2 - X_2^2) - (x_3^2 - x_4^2)(X_3^2 - X_4^2) - (x_5^2 - x_6^2)(X_5^2 - X_6^2), \\ q_3 &:= x_1x_2X_1X_2 - x_3x_4X_3X_4 - x_5x_6X_5X_6, \end{aligned}$$

([Bar87, Proposition 4.6], we used the formulas from [Bar87, p.68] to replace the λ_i, μ_i by the x_i , but notice that the factors ‘2’ in the formulas for $\lambda_i\mu_i$ should be omitted, so $\lambda_1\mu_1 = x_3^2 + x_4^2$ etc.). The 16 nodes of the Kummer surface are the orbit of x under the action of $T(2, 4)[2]$, that is, it is the set

$$\text{Nodes}(K_x) = \{p_{a,b,c,d} := (\tilde{\sigma}_1^a \tilde{\sigma}_2^{2b} \tilde{\tau}_1^c \tilde{\tau}_2^{2d})(x), \quad a, b, c, d \in \{0, 1\}\},$$

cf. Section 1.4.2. We considered the following six nodes:

$$p_{0,0,0,0}, \quad p_{0,0,1,1}, \quad p_{0,1,0,0}, \quad p_{0,1,1,0}, \quad p_{1,1,1,0}, \quad p_{1,1,1,1}.$$

For general $x \in \mathbb{P}_{QM}^1$ one finds that these six nodes span only a hyperplane H_x in \mathbb{P}^5 .

Using Magma [Magma] we found that over the quadratic extension of the function field $\mathbb{Q}(\zeta)(u)$ of \mathbb{P}_{QM}^1 (where $\zeta^4 = -1$ and $u = x/y$) defined by $w^2 = u^8 + 14u^4 + 1$, the intersection of H_x and K_x is reducible and consists of two rational curves of degree four, meeting in the 6 nodes.

We parametrize H_x by $t_1 p_{0,0,0,0} + \dots + t_5 p_{1,1,1,0}$. Then Magma shows that the rational function t_4/t_5 restricted to each of the two components is a generator of the function field of each of the two components. Thus t_4/t_5 provides a coordinate on each component and, for each component, we computed the value (in $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$) of the coordinate in the 6 nodes. The genus two curve $C = C_x$ is the double cover of \mathbb{P}^1 branched in these six points.

1.5.4 Invariants of genus two curves

A genus two curve over a field of characteristic 0 defines a homogeneous sextic polynomial in two variables, uniquely determined up to the action of $\text{Aut}(\mathbb{P}^1)$. In [Igu60, p.620], Igusa defines invariants A, B, C, D of a sextic and defines further invariants J_i , $i = 2, 4, 6, 10$, as follows [Igu60, p.621-622]:

$$\begin{aligned} J_2 &= 2^{-3}A, & J_4 &= 2^{-5}3^{-1}(4J_2^2 - B), \\ J_6 &= 2^{-6}3^{-2}(8J_2^3 - 160J_2J_4 - C), & J_{10} &= 2^{-12}D. \end{aligned}$$

In [Igu60, Theorem 6], Igusa showed that the moduli space of genus two curves over $\text{Spec}(\mathbb{Z})$ is a (singular) affine scheme which can be embedded in the affine space $\mathbb{A}_{\mathbb{Z}}^{10}$. Its restriction to $\text{Spec}(\mathbb{Z}[1/2])$ can be embedded into $\mathbb{A}_{\mathbb{Z}[1/2]}^8$ using the functions ([Igu60, p.642])

$$\begin{aligned} J_2^5 J_{10}^{-1}, & \quad J_2^3 J_4 J_{10}^{-1}, & J_2^3 J_4^2 J_{10}^{-1}, & \quad J_2^2 J_6 J_{10}^{-1}, \\ J_4 J_6 J_{10}^{-1}, & \quad J_2 J_6^3 J_{10}^{-2}, & J_4^5 J_{10}^{-2}, & \quad J_6^5 J_{10}^{-3}. \end{aligned}$$

From this one finds that over $\text{Spec}(\mathbb{Q})$ one can embed the moduli space into $\mathbb{A}_{\mathbb{Q}}^8$ using 8 functions i_1, \dots, i_8 as above but with J_2, \dots, J_{10} replaced by A, \dots, D . In case $A \neq 0$, one can use the three regular functions

$$j_1 := A^5/D, \quad j_2 := A^3B/D, \quad j_3 := A^2C/D$$

to express i_1, \dots, i_8 as

$$j_1, \quad j_2, \quad j_2^2/j_1, \quad j_3, \quad j_2j_4/j_1, \quad j_4^3/j_1, \quad j_2^5/j_1^3, \quad j_4^5/j_1^2.$$

Thus the open subset of the moduli space over \mathbb{Q} where $A \neq 0$ can be embedded in $\mathbb{A}_{\mathbb{Q}}^3$ using these three functions. In particular, two homogeneous sextic polynomials f, g with complex coefficients and with $A(f), A(g) \neq 0$ define isomorphic genus two curves over \mathbb{C} if and only if $j_i(f) = j_i(g)$ for $i = 1, 2, 3$ (see also [Mes91], [CQ05]).

1.5.5 Invariants of the curve C_x

With the Magma command ‘IgusaClebschInvariants’ we computed the invariants for each of the two genus curves which are the double covers of the two rational curves in $H_x \cap K_x$. They turn out to be isomorphic as expected from Rotger’s uniqueness result. We denote by C_x the corresponding genus two curve. For the general $x \in \mathbb{P}_{QM}^1$ the invariant $A = A(C_x)$ is non-zero and

$$j_1(C_x) = -3^5 2^{-5} \frac{(1 - 64G(x))^5}{G(x)^3}, \quad j_2(C_x) = 3^5 2^{-3} \frac{(1 - 64G(x))^3}{G(x)^2},$$

and

$$j_3(C_x) = 3^4 2^{-3} \frac{(1 - 64G(x))^2 (1 - 80G(x))}{G(x)^2}.$$

Notice that the invariants are rational functions in the \mathcal{S}_4 -invariant function $G = g_6^4/g_8^3$ on \mathbb{P}_{QM}^1 , as expected. Moreover, the $j_i(C_x)$ actually determine $G(x)$:

$$G(x) = \frac{(j_2(x)/j_3(x)) - 3}{80(j_2(x)/j_3(x)) - 192},$$

hence the classifying map from (an open subset of) $\mathbb{P}_{QM}^1/\mathcal{S}_4$ to the moduli space of genus two curves is a birational isomorphism onto its image.

1.5.6 The genus two curves from Hashimoto-Murabayashi

In [HM95, Theorem 1.3], Hashimoto and Murabayashi determine an explicit family of genus two curves $C_{s,t}$ whose Jacobians have quaternionic multiplication by the maximal order \mathcal{O}_6 . They are parametrized by the elliptic curve

$$E_{HM} : \quad g(t, s) = 4s^2 t^2 - s^2 + t^2 + 2 = 0.$$

Using the following rational functions on this curve:

$$P := -2(s + t), \quad R := -2(s - t), \quad Q := \frac{(1 + 2t^2)(11 - 28t^2 + 8t^4)}{3(1 - t^2)(1 - 4t^2)},$$

the genus two curve $C_{s,t}$ corresponding to the point $(s, t) \in E_{HM}$ is defined by the Weierstrass equation:

$$C_{s,t} : \quad Y^2 = X(X^4 - PX^3 + QX^2 - RX + 1).$$

By the unicity result from [Rot04, section 7] we know that this one parameter family of genus two curves should be the same as the one parametrized by \mathbb{P}_{QM}^1 . Indeed one has:

Proposition 1.5.2. *The genus two curve C_x defined by $x \in \mathbb{P}_{QM}^1$ is isomorphic to the curve $C_{s,t}$ if and only if $G(x) = H(t)$ (so the isomorphism class of $C_{s,t}$ does not depend on s) where*

$$H(t) := \frac{4(t-1)^2(t+1)^2(t^2+1/2)^4}{27((1-2t)(1+2t))^3}.$$

Proof. This follows from a direct Magma computation of the invariants j_i for the $C_{s,t}$. In particular, the classifying map of the Hashimoto-Murabayashi family has degree 12 on the t -line (and degree 6 on the $u := t^2$ -line), and this degree six cover is not Galois. \square

1.5.7 Special points

In Theorem 1.4.2 we observed that \mathcal{S}_4 acts on \mathbb{P}_{QM}^1 and has three orbits which have less than 24 elements, they are the zeroes of the polynomials g_d , of degree d , with $d = 6, 8, 12$. In case $d = 12$ one finds that for example $x = \zeta$ is a zero of g_{12} . The invariants $j_i(C_x)$ are the same as the invariants of the curve $C_{s,t}$ from [HM95] with $(t, s) = (0, \sqrt{2})$. In [HM95, Example 1.5] one finds that the Jacobian of this curve is isogenous to a product of two elliptic curves with complex multiplication by $\mathbb{Z}[\sqrt{-6}]$.

In case $d = 6, 8$ one finds that the invariants $j_i(C_x)$ are infinite, hence these points do not correspond to Jacobians of genus two curves but to products of two elliptic curves (with the product polarization). In case $g_6(x) = 0$ one finds that the intersection of the plane H_x with the Kummer surface K_x consists of four conics, each of which passes through four nodes (and there are now 8 nodes in $H_x \cap K_x$). The inverse image of each conic in the Abelian surface A_x is an elliptic curve which is isomorphic to $E_4 := \mathbb{C}/\mathbb{Z}[i]$, and one finds that $A_x \cong E_4 \times E_4$, but the $(1, 2)$ polarization is not the product polarization. The point $(t, s) = (\sqrt{-2}/2, \sqrt{2}/2) \in E_{HM}$ defines the same point in the Shimura curve $\mathbb{P}_{QM}^1/\mathcal{S}_4$ as the zeroes of g_6 . It corresponds to the degenerate curve $C_{t,s}$ in [HM95, Example 1.4], which has a normalization which is isomorphic to E_4 .

In case $d = 8$ one has $A_x \cong E_3 \times E_3$ and, with the $(1, 2)$ -polarization, it is the surface A_3 that we defined in Section 1.2.1. According to [Bar87, Theorem 4.9] a point $x \in M_{2,4}$ defines an Abelian surface A_x if and only if $r(x) \neq 0$ where $r = r_{12}r_{13}r_{23}$ is defined in [Bar87, Proposition 3.2] (the r_{jk} are polynomials in λ_i^2, μ_i^2 and these again can be represented by polynomials in the x_i , see [Bar87, p. 68]. One can choose these polynomials as follows:

$$\begin{aligned} r_{12} &= -4r_{13} = -4r_{23} = \\ &= 16(x_1x_6 - x_2x_5)(x_1x_6 + x_2x_5)(x_1x_5 - x_2x_6)(x_1x_5 + x_2x_6), \end{aligned}$$

and thus $r = 16r_{12}^3$. Restricting r to \mathbb{P}_{QM}^1 and pulling back along the parametrization to \mathbb{P}^1 , one finds that $r = cg_8^3$, where g_8 is as in Section

1.4.2 and c is a non-zero constant. More in general, we have the following result.

Proposition 1.5.3. *The image of the period matrices $\tau \in \mathbb{H}_2$ with $\tau_{12} = \tau_{21} = 0$ in $M_{2,4} \subset \mathbb{P}^5$ is the intersection of $M_{2,4}$ with the Segre threefold which is the image of the map*

$$S_{1,2} : \mathbb{P}^1 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^5, \quad ((u_0 : u_1), (w_0 : w_1, w_2)) \longrightarrow (x_1 : \dots : x_6)$$

where the coordinate functions are

$$\begin{aligned} x_1 &= u_0 w_0, & x_3 &= u_0 w_1, & x_5 &= u_0 w_2, \\ x_2 &= u_1 w_0, & x_4 &= u_1 w_1, & x_6 &= u_1 w_2. \end{aligned}$$

The image of $S_{1,2}$ intersects \mathbb{P}_{QM}^1 in two points which are zeroes of g_8 . Moreover, the surface $S_{1,2}(\mathbb{P}^1 \times \mathbb{P}^2) \cap M_{2,4}$ is an irreducible component of $(r = 0) \cap M_{2,4}$.

Proof. If $\tau_{12} = \tau_{21} = 0$, then by looking at the Fourier series which define the theta constants, one finds that $\vartheta_{[00]}^{ab}(\tau) = \vartheta_{[0]}^a(\tau_{11})\vartheta_{[0]}^b(\tau_{22})$. The definition of the x_i 's in terms of the standard delta functions in $V(2, 4)$, $u_n v_m = \vartheta_{[00]}^{ab}(\tau)$ with $(a, b) = (n/2, m/4)$ ([Bar87, p.53]), then shows that the map $\mathbb{H}_2 \rightarrow \mathbb{P}^5$ restricted to these period matrices is the composition of the map

$$\begin{aligned} \mathbb{H}_1 \times \mathbb{H}_1 &\longrightarrow \mathbb{P}^1 \times \mathbb{P}^2, \\ (\tau_1, \tau_2) &\longmapsto ((\vartheta_{[0]}^a(\tau_1) : \vartheta_{[0]}^b(\tau_1)), (\vartheta_{[0]}^a(\tau_2) + \vartheta_{[0]}^b(\tau_2) : \\ &\quad \vartheta_{[0]}^a(\tau_2) + \theta_{[0]}^c(\tau_2) : \theta_{[0]}^a(\tau_2) - \theta_{[0]}^b(\tau_2))) \end{aligned}$$

with the Segre map as above and $a, b, c = 1/4, 1/2, 3/4$ respectively.

The ideal of the image of $S_{1,2}$ is generated by three quadrics, restricting these to P_{QM}^1 one finds that the intersection of the image with P_{QM}^1 is defined by the quadratic polynomial $x^2 + (\zeta^2 - 1)xy + \zeta^2 y^2$, which is a factor of g_8 .

The factor $x_1 x_6 - x_2 x_5$ of r is in the ideal of $S_{1,2}(\mathbb{P}^1 \times \mathbb{P}^2)$, hence this surface is an irreducible component of $(r = 0) \cap M_{2,4}$. \square

Remark 1.5.3. The intersection of the image of $S_{1,2}$ with $M_{2,4}$, which is defined by $f_1 = f_2 = 0$ (cf. Section 1.4.1), is the image of the surface

$$\mathbb{P}^1 \times C_F, \quad (C \subset \mathbb{P}^1 \times \mathbb{P}^2), \quad C_F : w_0^4 - w_1^4 - w_2^4 = 0.$$

The curves \mathbb{P}^1 and C_F here are both elliptic modular curves (defined by the totally symmetric theta structures associated to the divisors $2O$ and $4O$, where O is the origin of the elliptic curve).

1.5.8 A Humbert surface

In section 1.5.3 we considered six nodes of the Kummer surface K_x ,

$$p_{0,0,0,0}, p_{0,0,1,1}, \dots, p_{1,1,1,1},$$

which had the property that for a general $x \in \mathbb{P}_{QM}^1$ these six nodes span only a hyperplane in \mathbb{P}^5 . For general $x \in M_{2,4}$ however these nodes do span all of \mathbb{P}^5 . They span at most a hyperplane if the determinant F of the 6×6 matrix whose rows are the homogeneous coordinates of the nodes, is equal to zero.

$$F = \det \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ -x_2 & x_1 & x_4 & -x_3 & x_6 & -x_5 \\ -x_2 & x_1 & -x_4 & x_3 & x_6 & -x_5 \\ x_1 & -x_2 & x_3 & -x_4 & -x_5 & x_6 \\ x_1 & x_2 & x_3 & x_4 & -x_5 & -x_6 \\ x_1 & -x_2 & -x_3 & x_4 & x_5 & -x_6 \end{pmatrix}.$$

Then F is a homogeneous polynomial of degree six in the coordinates of x which has 8 terms. Let D_F be the divisor in $M_{2,4}$ defined by $F = 0$, then \mathbb{P}_{QM}^1 is contained in (the support of) D_F . Magma shows that D_F has 12 irreducible components, the only one of these which contains \mathbb{P}_{QM}^1 is the surface $S_2 \subset \mathbb{P}^5$ defined by

$$S_2 : x_1^2 - x_2^2 - x_5^2 - x_6^2 = x_1x_2 - x_4^2 - x_5x_6 = x_3^2 - x_4^2 - 2x_5x_6 = 0.$$

Magma verified that S_2 is a smooth surface, hence it is a K3 surface.

Proposition 1.5.4. *The surface $S_2 \subset M_{2,4}$ parametrizes Abelian surfaces A with $\mathbb{Z}[\sqrt{2}] \subset \text{End}(A)$.*

Proof. For a general point x in S_2 , the hyperplane spanned by the six nodes intersects K_x in a one-dimensional subscheme which is the complete intersection of three quadrics and which has six nodes. The arithmetic genus of a smooth complete intersection of three quadrics in $H_x = \mathbb{P}^4$ is only five, hence this subscheme must be reducible. In the case $x \in \mathbb{P}_{QM}^1$, this subscheme is the union of two smooth rational curves of degree four intersecting transversally in the six nodes. Thus for general $x \in S_2$, the intersection must also consist of two such rational curves. Let $C \subset A_x$ be the genus two curve in the Abelian surface A_x defined by x which is the inverse image of one of these components. Then $C^2 = 2$ and $C \cdot \mathcal{L} = 4$, where \mathcal{L} defines the $(1, 2)$ -polarization. Now we apply [BL04, Proposition 5.2.3] to the endomorphism $f = \phi_C^{-1} \phi_{\mathcal{L}}$ of A_x defined by these polarizations. We find that the characteristic polynomial of f is $t^2 - 4t + 2$. As its roots are $2 \pm \sqrt{2}$, we conclude that $\mathbb{Z}[\sqrt{2}] \subset \text{End}(A_x)$. \square

Chapter 2

Cohomology of surfaces isogenous to a product

2.1 Basic definitions

2.1.1 Surfaces isogenous to a product

Definition 2.1.1 ([Cat00]). A smooth surface S is said to be isogenous to a product (of curves) if it is isomorphic to a quotient $\frac{C \times D}{G}$ where C and D are curves of genus at least one and G is a finite group acting freely on $C \times D$.

If the genus of both curves is greater or equal than two S is said to be isogenous to a higher product.

Let $S \cong \frac{C \times D}{G}$ be a surface isogenous to a product. The group G is identified with a subgroup of $\text{Aut}(C \times D)$ via the group action. We set

$$G^0 := G \cap (\text{Aut}(C) \times \text{Aut}(D)).$$

The group $\text{Aut}(C) \times \text{Aut}(D)$ is a normal subgroup of $\text{Aut}(C \times D)$ of index one or two, thus or $G = G^0$ or $[G : G^0] = 2$. In particular an element in the subgroup G^0 acts on each curve and diagonally on the product, conversely an element $g \in G$ but not in G^0 acts on the product interchanging factors.

Definition 2.1.2. Let S be a surface isogenous to a product. Then $\frac{C \times D}{G}$ is a minimal realization of S if $S \cong \frac{C \times D}{G}$ and G^0 acts faithfully on both curves.

Proposition 2.1.3 ([Cat00], Proposition 3.13). *Let S be a surface isogenous to a higher product. Then a minimal realization exists and it is unique.*

From now on whenever we refer to a surface S isogenous to an higher product we will always assume that it is given by its minimal realization. Let us fix some notation. We denote by $f : C \rightarrow C/G^0$ and $h : D \rightarrow D/G^0$

the morphisms induced by the action of G^0 on C and D respectively and by $\pi : C \times D \rightarrow S$ the étale covering given by the action of G on $C \times D$. Moreover we denote by $p_1 : C \times D \rightarrow C$ and by $p_2 : C \times D \rightarrow D$ the natural projections respectively on the first and on the second factor.

Proposition 2.1.4. *Let $S \cong \frac{C \times D}{G}$ be a surface isogenous to a higher product. Then:*

- S is of general type;
- S is minimal.

Proof. In this proof we will denote by $k(X)$ the Kodaira dimension of a surface X .

By hypothesis $g(C) \geq 2$ and $g(D) \geq 2$, then we have $k(C \times D) = 2$ by [Bea96, Proposition VII.4]. The map $\pi : C \times D \rightarrow S$ is an étale covering, hence $k(S) = k(C \times D)$ by [Bea96, Exercise VII.7]. Thus S is of general type.

The canonical divisors K_C and K_D of the curves C and D are ample, since $g(C) \geq 2$ and $g(D) \geq 2$. By the Künneth formula $K_{C \times D} = p_1^* K_C \otimes p_2^* K_D$ where $K_{C \times D}$ is the canonical divisor of the surface $C \times D$. Then also $K_{C \times D}$ is ample. As we have already observed $\pi : C \times D \rightarrow S$ is an étale covering, so $K_{C \times D}$ ample implies K_S ample. Hence S is a surface of general type with ample canonical divisor and then it is minimal. \square

Definition 2.1.5. Let $S \cong \frac{C \times D}{G}$ be a surface isogenous to a product. S is said to be of unmixed type if $G = G^0$, of mixed type otherwise.

Proposition 2.1.6. *Let $S \cong \frac{C \times D}{G}$ be a surface isogenous to a higher product of unmixed type. Then S admits two isotrivial fibrations whose smooth fibers are isomorphic to C in one case and to D in the other.*

Proof. The fibrations $\tilde{f} : S \rightarrow C/G$ and $\tilde{h} : S \rightarrow D/G$ are defined by the following diagrams:

$$\begin{array}{ccc} C \times D & \xrightarrow{\pi} & S \\ \downarrow p_1 & & \downarrow \tilde{f} \\ C & \xrightarrow{f} & C/G \end{array} \quad \begin{array}{ccc} C \times D & \xrightarrow{\pi} & S \\ \downarrow p_2 & & \downarrow \tilde{h} \\ D & \xrightarrow{h} & D/G \end{array}$$

where p_1, p_2, f, h and π are defined above. It is easy to check that \tilde{f} and \tilde{h} are well-defined.

Consider a point $x_0 \in C$ which is not a ramification point for f and $x := f(x_0)$. Then $p_1^{-1}f^{-1}(x)$ consists of exactly $|G|$ copies of D :

$$p_1^{-1}f^{-1}(x) = \{(gx_0, D) \in C \times D : g \in G\}.$$

Thus $\tilde{f}^{-1}(x) = \pi p_1^{-1} f^{-1}(x)$ is isomorphic to D , because G acts faithfully on both curves. Similarly we can prove that the fiber of $\tilde{h} : S \rightarrow D/G$ is isomorphic to C . \square

Proposition 2.1.7. *Let $S \cong \frac{C \times D}{G}$ be a surface isogenous to a product. Then the following equalities hold:*

- $\chi(\mathcal{O}_S) = \frac{(g(C)-1)(g(D)-1)}{|G|};$
- $e(S) = 4\chi(\mathcal{O}_S) = \frac{4(g(C)-1)(g(D)-1)}{|G|};$
- $K_S^2 = 8\chi(\mathcal{O}_S) = \frac{8(g(C)-1)(g(D)-1)}{|G|}.$

Proof. By the Künneth formula we get $q(C \times D) = g(C) + g(D)$ and $p_g(C \times D) = g(C)g(D)$. Then

$$\chi(\mathcal{O}_{C \times D}) = 1 - q(C \times D) + p_g(C \times D) = (g(C) - 1)(g(D) - 1).$$

Moreover we have $h^{1,1} = 2 + 2g(C)g(D)$ and then

$$e(C \times D) = 4(g(C) - 1)(g(D) - 1) = 4\chi(\mathcal{O}_S).$$

The map $\pi : C \times D \rightarrow S$ is an étale covering of degree $|G|$, then by [Bea96, Lemma VI.3] we obtain:

$$\chi(\mathcal{O}_S) = \frac{\chi(\mathcal{O}_{C \times D})}{|G|}, \quad e(S) = \frac{e(C \times D)}{|G|}.$$

At last we apply Noether's formula to compute K_S^2 :

$$K_S^2 = 12\chi(\mathcal{O}_S) - e(S) = 8\chi(\mathcal{O}_S).$$

\square

Lemma 2.1.8 ([Bea96], Lemma VI,11). *Let X be a smooth projective variety and G a finite subgroup of $\text{Aut}(X)$. Let $\pi : X \rightarrow Y = X/G$ denote the natural projection, and assume that Y is smooth. Then $\pi^* : H^0(Y, \Omega_Y^p) \rightarrow H^0(X, \Omega_X^p)^G$ is an isomorphism for all p .*

Proposition 2.1.9. *Let $S \cong \frac{C \times D}{G}$ be a surface isogenous to a product of unmixed type. Then*

$$q(S) = g\left(\frac{C}{G}\right) + g\left(\frac{D}{G}\right). \quad (2.1)$$

Proof. The result follows from Lemma 2.1.8 and the Künneth formula. \square

Proposition 2.1.10. *Let S be a surface isogenous to a higher product with $\chi(\mathcal{O}_S) = 2$ and $q(S) = 0$. Then the Hodge diamond is fixed:*

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 4 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \quad (2.2)$$

Proof. By hypothesis we have $h^{1,0}(S) = 0$ and $h^{2,0} = \chi(\mathcal{O}_S) - 1 = 1$: we just have to compute $h^{1,1}(S)$. By Proposition 2.1.7 $e(S) = 4\chi(\mathcal{O}_S) = 8$ and then

$$h^{1,1}(S) = e(S) - 2 + 4q(S) - 2p_g(S) = 4$$

□

Surfaces isogenous to a higher product of unmixed type with $\chi(\mathcal{O}_S) = 2$ and $q = 0$ have been studied and classified in [Gle13]: see section 2.3.2 for the details.

Definition 2.1.11 ([Cat00]). A Beauville surface is a rigid surface which is isogenous to a product.

2.1.2 Spherical system of generators

Definition 2.1.12. Let G be a group and $r \in \mathbb{N}$ with $r \geq 2$. An r -tuple $T = [g_1, \dots, g_r]$ of elements in G is called spherical system of generators of G if g_1, \dots, g_r is a system of generators of G and we have $g_1 \cdot \dots \cdot g_r = \text{Id}_G$. We call $\ell(T) := r$ length of S .

Definition 2.1.13. Let $A = [m_1, \dots, m_r] \in \mathbb{N}^r$ be an r -tuple of natural numbers $2 \leq m_1 \leq \dots \leq m_r$. A spherical system of generators $T = [g_1, \dots, g_r]$ is said to be of type $A = [m_1, \dots, m_r]$ if there is a permutation $\tau \in \mathcal{S}_r$ such that $\text{ord}(g_i) = m_{\tau(i)}$, for $i = 1, \dots, r$.

Let C be a curve and let $G \leq \text{Aut}(C)$ be a finite group such that $C/G \cong \mathbb{P}^1$. We denote by $f : C \rightarrow \mathbb{P}^1$ the quotient map induced by G . Let $B := \{b_1, \dots, b_r\}$ be the set of branch points on \mathbb{P}^1 and let $q \in \mathbb{P}^1$ be a point not in B . Then

$$\pi_1(\mathbb{P}^1 - B, q) = \langle \gamma_1, \dots, \gamma_r \rangle,$$

where γ_i is a simple counterclockwise loop around p_i .

Let $\psi : \pi_1(\mathbb{P}^1 - B, q) \rightarrow G$ be a monodromy representation. We observe that, by construction, $[\psi(\gamma_1), \dots, \psi(\gamma_r)]$ is a spherical system of generators.

By the Riemann Existence Theorem the correspondence above works also in the opposite sense:

Proposition 2.1.14. *Let G be a finite group and $B = \{b_1, \dots, b_r\} \subset \mathbb{P}^1$. Then there is a correspondence between:*

- *Spherical system of generators T of G with length $\ell(T) = r$;*
- *Galois covering $f : C \rightarrow \mathbb{P}^1$ with branch points B .*

Proof. It follows from [Mir95, Section III.3 and III.4]. \square

Remark 2.1.1. The curve C is completely determined by the branch points B and by the spherical system of generators T . In particular the genus can be computed using the Riemann-Hurwitz formula:

$$g(C) = 1 - d + \sum_{i=1}^r \frac{d}{2m_i} (m_i - 1)$$

where $A = [m_1, \dots, m_r]$ is the type of T .

Remark 2.1.2. The correspondence of Proposition 2.1.14 is not one-to-one: indeed distinct spherical systems of generators could determine the same covering.

For example let $T_1 = [g_1, \dots, g_r]$ be a spherical system of generators of G of type A and let $h \in G$. Consider $T_2 = [g_1^h, \dots, g_r^h]$ where $g^h = h^{-1}gh$: T_2 is a spherical system of generators of type A and determines an isomorphic covering. In particular T_2 determines exactly the same covering, not only an isomorphic one, and it corresponds to a different choice of the monodromy representation.

Let $S = \frac{C \times D}{G}$ be a surface isogenous to a higher product of unmixed type with $q(S) = 0$. Then by Proposition 2.1.9 we get two ramified coverings of the sphere $f : C \rightarrow \mathbb{P}^1$ and $h : D \rightarrow \mathbb{P}^1$. Notice that, from a topological point of view, the surface S is determined by f and h under the further condition that the group G acts freely on the product $C \times D$.

Definition 2.1.15. Let $T = [g_1, \dots, g_r]$ be a spherical system of generators of G . We denote by $\Sigma(T)$ the union of all conjugates of the cyclic subgroups generated by the elements g_1, \dots, g_r :

$$\Sigma(T) := \Sigma([g_1, \dots, g_r]) = \bigcup_{g \in G} \bigcup_{j=0}^{\infty} \bigcup_{i=1}^r \{g \cdot g_i^j g^{-1}\}$$

A pair of spherical systems of generators (T_1, T_2) of G is called disjoint if

$$\Sigma(T_1) \cap \Sigma(T_2) = \{Id_G\}$$

Proposition 2.1.16. *Let T_1 and T_2 be two spherical systems of generators of G and consider the corresponding coverings $f : C \rightarrow \mathbb{P}^1$ and $h : D \rightarrow \mathbb{P}^1$. Let $\pi : C \times D \rightarrow \frac{C \times D}{G}$ be the induced covering where G acts on the product via the diagonal action. Then the following conditions are equivalent:*

- π is an étale covering, i.e. the action of G is free;
- (T_1, T_2) is a disjoint pair of spherical systems of generators of G .

Proof. We observe that an element $g \in G$ fixes a point in C if and only if $g \in \Sigma(T_1)$ and it fixes a point in D if and only if $g \in \Sigma(T_2)$. Then g fixes a point in $C \times D$ if and only if $g \in \Sigma(T_1) \cap \Sigma(T_2)$. \square

Definition 2.1.17. An unmixed ramification structure for G is a disjoint pair of spherical system of generators (T_1, T_2) of G .

Let $A_1 = [m_{(1,1)}, \dots, m_{(1,r_1)}]$ and $A_2 = [m_{(2,1)}, \dots, m_{(2,r_2)}]$ be respectively a r_1 -tuple and a r_2 -tuple of natural numbers with $2 \leq m_{(1,1)} \leq \dots \leq m_{(1,r_1)}$ and $2 \leq m_{(2,1)} \leq \dots \leq m_{(2,r_2)}$. We say that the unmixed ramification structure (T_1, T_2) is of type (A_1, A_2) if T_1 is of type A_1 and T_2 is of type A_2 .

Putting together Proposition 2.1.14 and Proposition 2.1.16 we get a correspondence between unmixed ramification structures and surfaces isogenous to a product of unmixed type. As already observed, this correspondence is not one-to-one, but it works well in one direction: given an unmixed ramification structure it is uniquely defined a surface isogenous to a product of unmixed type.

Lemma 2.1.18. A surface $S = \frac{C \times D}{G}$ isogenous to a higher product of unmixed type is a Beauville surface if and only if $C/G \cong D/G \cong \mathbb{P}^1$ and the morphisms $f : C \rightarrow C/G$ and $h : D \rightarrow D/G$ have both three branch points.

Proof. Up to compose with Möbius transformations, we can assume that $f : C \rightarrow \mathbb{P}^1$ and $h : D \rightarrow \mathbb{P}^1$ ramify over three fixed points (for example $\{0, 1, \infty\}$). Now the proof follows by Proposition 2.1.14. \square

2.2 Group representations

In this section we recall some basic facts about irreducible rational representations of a finite group G , in particular when G acts on a rational Hodge structure. In this section we do not prove almost anything since all the results are very well-known. See [Ser77] for a reference about rational representations, [BL04, Section 13.6] about the group algebra decomposition, [Voi02] and [vG00] about rational Hodge structures.

2.2.1 Irreducible rational representation

Let G be a finite group of order N . A complex representation $\rho : G \rightarrow GL(V)$ can be decomposed as sum of irreducible complex representations. G has exactly m irreducible complex representations where m is the number of conjugacy classes in G . We will write

$$\rho = \bigoplus_{i=1}^m n_{\rho}(\rho_i) \rho_i,$$

where $\rho_i : G \rightarrow GL(V_i)$, $i = 1, \dots, m$ are the irreducible complex representations of G and $n_\rho(\rho_i)$ is the multiplicity of ρ_i in ρ . We always denote by ρ_1 the trivial representation.

The complex representation $\rho : G \rightarrow GL(V)$ is determined by its character $\chi_\rho : G \rightarrow \mathbb{C}$ defined by $\chi_\rho(g) := \text{tr}(\rho(g))$ for all $g \in G$. Indeed the following relation holds:

$$\langle \rho, \rho_i \rangle = \frac{1}{N} \sum_{g \in G} \overline{\chi_\rho(g)} \chi_i(g) = n_\rho(\rho_i), \quad (2.3)$$

where $\chi_i : G \rightarrow \mathbb{C}$ is the character of the irreducible complex representation $\rho_i : G \rightarrow GL(V_i)$.

The character field K_ρ associated to ρ is the field $\mathbb{Q}(\chi_\rho(g))_{g \in G}$. As $\rho(g) \in GL(V)$ has finite order, its eigenvalues are roots of unity, hence K_ρ is a subfield of $\mathbb{Q}(\xi_N)$ where ξ_N is a primitive N -th root of unity.

Given an algebraic extension L of \mathbb{Q} we denote with $Gal(L/\mathbb{Q})$ the Galois group of the field L , i.e. the group of all automorphisms of L fixing \mathbb{Q} pointwise. We will use the shorter notation Gal_N instead of $Gal(\mathbb{Q}(\xi_N)/\mathbb{Q})$.

Proposition 2.2.1. *Let G be a finite group of order N and let $\rho_i : G \rightarrow GL(V_i)$ be an irreducible complex representation of G with associated character field K_i . For every $\sigma \in Gal(K_i/\mathbb{Q})$ there exists a unique irreducible complex representation $\rho_j : G \rightarrow GL(V_j)$ with character $\chi_j = \sigma(\chi_i)$. Thus for σ, ρ_i and ρ_j as above we set $\sigma(\rho_i) = \rho_j$.*

In the same way we can define an action of the whole group Gal_N on the irreducible complex representations.

Definition 2.2.2. Let $\rho_i : G \rightarrow GL(V_i)$ be an irreducible complex representation of G with character field K_i . The dual representation of ρ_i is the irreducible complex representation $\overline{\rho_i} := \tilde{\sigma}(\rho_i)$ where $\tilde{\sigma}$ is the complex conjugation.

We say that ρ_i is self-dual if $\rho_i = \overline{\rho_i}$ or, equivalently, if $K_i \subseteq \mathbb{R}$.

The action of Gal_N splits the set of the irreducible complex representations into distinct orbits such that if two irreducible complex representations ρ_i and ρ_j are in the same orbit then $K_i = K_j$.

Proposition 2.2.3. *Let G be a finite group of order N and let $\tau : G \rightarrow GL(W)$ be an irreducible rational representation. Then there is a unique Gal_N -orbit of irreducible complex representations $\{\sigma(\rho_i)\}_{\sigma \in Gal(K_i/\mathbb{Q})}$, $\rho_i : G \rightarrow GL(V_i)$ and a positive integer s , called Schur index of ρ_i , such that*

$$\tau_{\mathbb{C}} := \tau \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma \in Gal(K_i/\mathbb{Q})} s \cdot \sigma(\rho_i). \quad (2.4)$$

Conversely each irreducible complex representation $\rho_i : G \rightarrow GL(V_i)$ determines a unique irreducible rational representation $\tau : G \rightarrow GL(W)$ such that the equality (2.4) holds.

Corollary 2.2.3.1. *The following equality holds:*

$$\dim_{\mathbb{Q}} W = \dim_{\mathbb{C}} W_{\mathbb{C}} = s \cdot \dim_{\mathbb{C}} V_i \cdot [K_i : \mathbb{Q}]$$

Corollary 2.2.3.2. *Let $\rho : G \rightarrow GL(V)$ be a self-dual complex representation such that $K_{\rho} = \mathbb{Q}$. Then there exists a rational representation $\tau : G \rightarrow GL(W)$ and a positive integer s such that $\tau \otimes_{\mathbb{Q}} \mathbb{C} = s \cdot \rho$.*

Any rational representation $\tau : G \rightarrow GL(W)$ can be decomposed as sum of irreducible rational representations, exactly as it happens for the complex ones. We will write

$$\tau = \bigoplus_{j=1}^t n_{\tau}(\tau_j) \tau_j,$$

where $\tau_j : G \rightarrow GL(W_j)$, $j = 1, \dots, t$ are the irreducible rational representations of G and $n_{\tau}(\tau_j)$ is the multiplicity of τ_j in τ . As in the complex case, we denote by τ_1 the trivial representation.

Remark 2.2.1. For the rational representations there is not an analogue of the relation (2.3). Usually, in order to decompose a rational representation $\tau : G \rightarrow GL(W)$, we decompose $\tau_{\mathbb{C}}$ into irreducible complex representations and then we use Proposition 2.2.3.

Example 2.2.1. Let G be the cyclic group $\mathbb{Z}_3 := \mathbb{Z}/3\mathbb{Z}$. G is an abelian group and its character table is

	1	2	3
χ_1	1	1	1
χ_2	1	ξ_3	ξ_3^2
χ_3	1	ξ_3^2	ξ_3

\mathbb{Z}_3 has 3 irreducible complex representations ρ_1, ρ_2 and ρ_3 where ρ_i is the representation associated to the character χ_i .

There are exactly two orbits in the set of irreducible complex representations. Then G has only 2 irreducible rational representations τ_1 and τ_2 such that $\tau_1 \otimes \mathbb{C} = \rho_1$ and $\tau_2 \otimes \mathbb{C} = \rho_2 \oplus \rho_3$.

Example 2.2.2. Consider the quaternion group Q_8 :

$$Q_8 = \langle -1, i, j, k \mid (-1)^2 = 1, i^2 = j^2 = k^2 = ijk = -1 \rangle.$$

Its character table is:

	1	-1	$\pm i$	$\pm j$	$\pm k$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	-1	1
χ_4	1	1	-1	1	-1
χ_5	2	-2	0	0	0

In this case any irreducible complex representation ρ_i , $i = 1, \dots, 5$, has character field $K_i = \mathbb{Q}$ and then defines a different Galois orbit. So G has 5 irreducible rational representations τ_j , $j = 1, \dots, 5$.

The Schur index of the first four representations has to be one, because the Schur index divides the dimension of the representation (see [Ser77]); thus for $i = 1, 2, 3, 4$ we get $\tau_i \otimes \mathbb{C} = \rho_i$. In particular, as the dimension of the representations is one, we have $\rho_i = \chi_i$.

Now we will prove that the Schur index of ρ_5 is two, i.e. that there is not a rational representation of dimension two whose complexification is ρ_5 .

Let us suppose by contradiction that there exists a rational representation $\tau : G \rightarrow GL(2, \mathbb{Q})$ with character χ_5 . Notice that $\tau(-1)$ is a matrix of order 2 and trace -2 : we easily deduce that

$$\tau(-1) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now we have to determine $\tau(i)$ and $\tau(j)$: they are matrices of order 4 with trace 0. We get that they are of the form

$$\tau(i) = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \quad (2.5)$$

$$a^2 + bc = -1 \quad (2.6)$$

with $a, b, c \in \mathbb{Q}$. From this last equation we obtain that $b \neq 0$ and $c \neq 0$: thus for each non zero $v \in \mathbb{R}^2$, the vectors v and $\tau(i)v$ are linearly independent. So, up to a change of basis, we can suppose without loss of generality that $\tau(i)$ is

$$\tau(i) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then on the same basis $\tau(j)$ is a matrix of the form (2.5) such that $\tau(i)\tau(j) = -\tau(j)\tau(i)$. But this last condition is equivalent to $b = c$ and so the equation (2.6) becomes $a^2 + b^2 = -1$ with $a, b \in \mathbb{Q}$. A contradiction.

On the contrary we can construct a representation τ with $\tau \otimes \mathbb{C} = 2\rho_5$ setting:

$$\tau(i) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \tau(j) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

2.2.2 The group algebra decomposition

Let K be a field and G a finite group. The group algebra $K[G]$ is the ring of the K -linear combinations of elements in G , where the product is obtained by linearly extending the group operation. If we have a K -representation

$\tau : G \rightarrow GL(W)$ we can extend this to a ring homomorphism $\tilde{\tau} : K[G] \rightarrow \text{End}(W)$ setting:

$$\tilde{\tau} \left(\sum a_i g_i \right) = \sum a_i \tau(g_i)$$

where $a_i \in K$ and $g_i \in G$.

Let G be a finite group with irreducible complex representations $\rho_i : G \rightarrow GL(V_i)$, $i = 1, \dots, m$ where m is the number of conjugacy classes of G . Consider the following elements in $\mathbb{C}[G]$:

$$p_i = \frac{\dim(V_i)}{\#G} \sum_{g \in G} \chi_i(g) g$$

where χ_i is the character of ρ_i .

Proposition 2.2.4. *The elements p_1, \dots, p_m are central idempotents in the group algebra $\mathbb{C}[G]$, i.e. $p_i^2 = p_i$ and $p_i g = g p_i$ for all $g \in G$. Moreover we get:*

$$\tilde{\rho}_i(p_j) = \begin{cases} Id_{V_i} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (2.7)$$

Let us consider the group algebra $\mathbb{Q}(\xi_N)[G]$ where N is the order of G and notice that $p_i \in \mathbb{Q}(\xi_N)[G]$ for all $i = 1, \dots, m$. There is a natural action of the Galois group Gal_N on $\mathbb{Q}(\xi_N)[G]$ defined by

$$\sigma \left(\sum a_j g_j \right) = \sum \sigma(a_j) g_j$$

where $\sigma \in Gal_N$.

This action agrees with the action defined in Proposition 2.2.1: $\sigma(\rho_i) = \rho_j$ if and only if $\sigma(p_i) = p_j$.

Let $\tau_j : G \rightarrow GL(W_j)$ be an irreducible rational representation. By Proposition 2.2.3 there exists an irreducible complex representation $\rho_i : G \rightarrow GL(V_i)$ such that

$$\tau_j = \bigoplus_{\sigma \in Gal(K_i/\mathbb{Q})} s \cdot \sigma(\rho_i).$$

We define

$$q_j = \sum_{\sigma \in Gal(K_i/\mathbb{Q})} \sigma(p_i).$$

Proposition 2.2.5. *Let G be a finite group and let $\tau_j : G \rightarrow GL(W_j)$, $j = 1, \dots, t$ be its irreducible rational representations. Then $q_j \in \mathbb{Q}[G]$ for all $j = 1, \dots, t$ and*

$$\tilde{\tau}_i(q_j) = \begin{cases} Id_{W_i} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (2.8)$$

Proof. By definition $q_j \in \mathbb{Q}(\xi_N)[G]$. For all $g \in G$ the coefficient of g in q_j is given by the equation

$$c_g := \frac{\dim(V_i)}{\#G} \sum_{\sigma \in \text{Gal}(K_i/\mathbb{Q})} \sigma(\chi_i(g))$$

By hypothesis $\chi_i(g) \in K_i$ and then $c_g \in \mathbb{Q}$ for all $g \in G$.

In order to prove equation (2.8) we have to complexify it and compare with equation (2.7). \square

Proposition 2.2.6. *Let $\tau : G \rightarrow GL(W)$ be a rational representation. We define $A_j = \text{Im}\{\tilde{\tau}(q_j) : W \rightarrow W\}$. Then*

- A_j is a rational subrepresentation and $\tau|_{A_j} = m_\tau(\tau_j)\tau_j$;
- $W = \bigoplus_{j=1}^t A_j$.

Definition 2.2.7. Let $\tau : G \rightarrow GL(W)$ be a rational representation. We call A_j the isotypical component related to the representation τ_j and we call $W = \bigoplus_{j=1}^t A_j$ isotypical decomposition of τ .

Proposition 2.2.8. *Let G be a finite group and $\rho_i : G \rightarrow GL(V_i)$ $i = 1, \dots, m$ its irreducible complex representations. We set $\rho = \bigoplus_{i=1}^m \rho_i : G \rightarrow GL(V)$. Then $\tilde{\rho} : \mathbb{C}[G] \rightarrow \bigoplus_{i=1}^m \text{End}(V_i)$ is an algebra isomorphism.*

Proof. Notice that the regular representation of G is injective and is given by $\bigoplus n_i \rho_i$, hence also ρ is injective. By Schur's orthogonality relations we have $\dim_{\mathbb{C}} \mathbb{C}[G] = \#G = \sum (\dim(V_i))^2 = \dim_{\mathbb{C}} \text{End}(V_i)$, hence $\tilde{\rho}$ is also surjective. \square

Remark 2.2.2. A similar result can not hold in general for rational representations, since the algebras $\mathbb{Q}[G]$ and $\bigoplus_{j=1}^t \text{End}(W_j)$ need not have the same dimension.

In order to avoid indices we work on a single irreducible rational representation $\tau : G \rightarrow GL(W)$. Consider $\mathbb{D} := \text{End}_G(W)$, the algebra of G -equivariant maps on W :

$$\mathbb{D} = \text{End}_G(W) = \{f \in \text{End}(W) : \tau(g)f = f\tau(g) \forall g \in G\}.$$

The kernel of any element $f \in \mathbb{D}$ is a subrepresentation of W , hence, as W is irreducible, all $f \in \mathbb{D}$ must be isomorphisms of W and then \mathbb{D} is a skew-field (or a division algebra). We consider W as a left vector space over \mathbb{D} , then choosing a basis we get:

$$W \cong \mathbb{D}^k$$

where $k = \dim_{\mathbb{D}}(W)$.

Suppose $\tau_{\mathbb{C}} = \oplus_{\sigma \in \text{Gal}(K_i)} s \cdot \sigma(\rho_i)$, where $\rho_i : G \rightarrow GL(V_i)$ is an irreducible complex representation and so $\text{End}_G(W_{\mathbb{C}}) = \oplus_{\sigma \in \text{Gal}(K_i)} \text{End}_G(V_i^{\oplus s})$. Then:

$$\begin{aligned} \dim_{\mathbb{Q}} W &= \dim_{\mathbb{C}} W_{\mathbb{C}} = s \cdot \dim_{\mathbb{C}}(V_i) \cdot [K_i : \mathbb{Q}], \\ \dim_{\mathbb{Q}} \mathbb{D} &= \dim_{\mathbb{C}} \mathbb{D}_{\mathbb{C}} = [K_i : \mathbb{Q}] \cdot \dim(\text{End}_G(V_i^{\oplus s})) = [K_i : \mathbb{Q}] \cdot s^2, \\ \dim_{\mathbb{D}} W &= k = \frac{[K_i : \mathbb{Q}] \dim_{\mathbb{C}}(V_i) \cdot s}{[K_i : \mathbb{Q}] \cdot s^2} = \frac{\dim_{\mathbb{C}}(V_i)}{s}. \end{aligned}$$

Recall that the Schur index s is always a divisor of the dimension of the representation and so $k \in \mathbb{N}$. By definition of \mathbb{D} , $\tau(g)$ commutes with \mathbb{D} for all $g \in G$ and so the image of $\tilde{\tau}$ lies in $\text{End}_{\mathbb{D}}(W)$. Moreover we observe that

$$\dim_{\mathbb{Q}}(\text{End}_{\mathbb{D}}(W)) = \dim_{\mathbb{Q}} \mathbb{D} \cdot \dim_{\mathbb{D}} \text{End}_{\mathbb{D}}(W) = (\dim_{\mathbb{C}}(V_i))^2 \cdot [K_i : \mathbb{Q}]. \quad (2.9)$$

Proposition 2.2.9. *Let G be a finite group and $\tau_j : G \rightarrow GL(W_j)$, $j = 1, \dots, t$, its irreducible rational representations. We set $\mathbb{D}_j = \text{End}_G(W_j)$ and $\tau = \oplus_{j=1}^t \tau_j : G \rightarrow GL(W)$. Then $\tilde{\tau} : \mathbb{Q}[G] \rightarrow \oplus_{j=1}^t \text{End}_{\mathbb{D}_j}(W_j)$ is an algebra isomorphism.*

Proof. From Proposition 2.2.8 we get the injectivity. Then it is enough to prove that the two algebras have the same dimension. Of course $\dim \mathbb{Q}[G] = \#G$. Now from equation (2.9) we get

$$\dim_{\mathbb{Q}} \left(\oplus_{j=1}^t \text{End}_{\mathbb{D}_j}(W_j) \right) = \oplus_{i=1}^m (\dim_{\mathbb{C}}(V_i))^2 = \#G.$$

□

By choosing a \mathbb{D} -basis of W we identify $\text{End}_{\mathbb{D}}(W)$ with the algebra $\text{Mat}(k, \mathbb{D})$ of matrices $k \times k$ with coefficients in \mathbb{D} . In particular in $\text{Mat}(k, \mathbb{D})$ we have matrices E_i with 1 at (i, i) and zero elsewhere. Then by the proposition above we are able to find k idempotents w_1, \dots, w_k in $\mathbb{Q}[G]$ such that $\tilde{\tau}(w_i) = E_i$.

Remark 2.2.3. These elements w_1, \dots, w_k are not unique, since they depend on the choice of a \mathbb{D} -basis.

This construction holds for all the irreducible rational representations. Given an irreducible rational representation $\tau_j : G \rightarrow GL(W_j)$ we denote by $w_{j,1}, \dots, w_{j,k_j}$ idempotent elements of $\mathbb{Q}[G]$ constructed as above.

Proposition 2.2.10. *Let $\tau : G \rightarrow GL(W)$ be a rational representation and let A_1, \dots, A_t be the isotypical components related to the irreducible rational representations of G . For all $j \in 1, \dots, t$ we define $B_j = \text{Im}\{\tilde{\tau}(w_{j,1}) : W \rightarrow W\}$. Then $A_j \cong B_j^{\oplus k_j}$ for all j , $j = 1, \dots, t$, $k_j = \dim_{\mathbb{D}_j} W_j$.*

Proof. By construction $w_{j,1} + \dots + w_{j,k_j} = q_j$ for all $j = 1, \dots, t$. Since $\tilde{\tau}(q_j)$ acts as the identity on A_j we get a decomposition:

$$A_j = \text{Im}\{\tilde{t}(w_{j,1})\} \oplus \dots \oplus \text{Im}\{\tilde{t}(w_{j,k_j})\}.$$

Fix a \mathbb{D}_j -basis of W_j and consider in $\text{End}_{\mathbb{D}_j}(W_j) \cong \text{Mat}(k_j, \mathbb{D}_j)$ the matrices M_i with 1 at $(i, 1)$ and zero elsewhere. These matrices provide isomorphisms between $B_j = \text{Im}\{\tilde{t}(w_{j,1})\}$ and $\text{Im}\{\tilde{t}(w_{j,i})\}$ for all $i = 2, \dots, k_j$. \square

Definition 2.2.11. Let $\tau : G \rightarrow GL(W)$ be a rational representation. We call B_j the isogenous component related to the representation τ_j and we call $W \cong \oplus_{j=1}^t B_j^{\oplus k_j}$ the isogenous decomposition of τ .

Remark 2.2.4. Unlike the isotypical components A_j , the isogenous components B_j are not G -subrepresentations. Indeed, as observed in the proof of the Proposition 2.2.10, the group algebra $\mathbb{Q}[G]$ interchanges the isogenous components.

Example 2.2.3. Let G be the group \mathbb{Z}_3 , studied in Example 2.2.1. \mathbb{Z}_3 has two irreducible rational representations: τ_1 (the trivial one) and τ_2 . Since τ_1 is an absolutely irreducible representation $\mathbb{D}_1 \cong \mathbb{Q}$. Instead the skew-field \mathbb{D}_2 has \mathbb{Q} -dimension 2 and so it is a field. With some computation we can prove

$$\mathbb{D}_5 \cong \mathbb{Q}(\xi_6)$$

where ξ_6 is a primitive 6-root of unity.

Example 2.2.4. Let Q_8 be the quaternion group defined in Example 2.2.2: we assume the same notations used there. The group has 5 irreducible rational representation. τ_1, τ_2, τ_3 and τ_4 are absolutely irreducible representations and so

$$\mathbb{D}_1 \cong \mathbb{D}_2 \cong \mathbb{D}_3 \cong \mathbb{D}_4 \cong \mathbb{Q}.$$

Consider now the representation τ_5 . Using the formulas above we observe that \mathbb{D}_5 has \mathbb{Q} -dimension 4. With some extra computation we can prove that

$$\mathbb{D}_5 \cong \frac{(-1, -1)}{\mathbb{Q}}.$$

2.2.3 Rational Hodge structures

Definition 2.2.12 ([vG00]). A rational Hodge structure of weight k is given by a rational vector space W together with a decomposition

$$W_{\mathbb{C}} = W \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=k} W^{p,q}$$

satisfying $W^{p,q} = \overline{W^{q,p}}$. Here the complex conjugation on $W_{\mathbb{C}}$ is given by $\overline{w \otimes z} = w \otimes \bar{z}$ for $w \in W$ and $z \in \mathbb{C}$.

Proposition 2.2.13. *There is a bijection between rational Hodge structures of weight k on a rational vector space W and algebraic representations $h : \mathbb{C}^* \rightarrow GL(W_{\mathbb{R}})$ with $h(t) = t^k$ for $t \in \mathbb{R}$.*

Proof. Proved in [vG00, Proposition 1.4] □

Example 2.2.5 (Tate Hodge structure). Consider on \mathbb{Q} the decomposition

$$(\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{C})^{p,q} = \mathbb{C}^{p,q} = \begin{cases} \mathbb{C} & \text{if } p = q = -n, \\ 0 & \text{otherwise.} \end{cases}$$

where $n \in \mathbb{Z}$. We denote this rational Hodge structure of weight $-2n$, called Tate Hodge structure, by $\mathbb{Q}(n)$.

It corresponds to the representation

$$h_n : \mathbb{C}^* \rightarrow \mathbb{R}^*, \quad z \mapsto (z\bar{z})^{-n}.$$

Example 2.2.6. Let W be a rational Hodge structure of weight k . We denote by $W(n)$ the rational Hodge structure of weight $k - 2n$ given by $W \otimes_{\mathbb{Q}} \mathbb{Q}(n)$.

Lemma 2.2.14. *Let W be a rational vector space. Then there is a one-to-one correspondence between:*

- *rational Hodge structures of weight 1 on W and*
- *almost complex structures on W , i.e. automorphisms $J : W_{\mathbb{R}} \rightarrow W_{\mathbb{R}}$ of order 4 such that $J^2 = -Id$.*

Proof. Let J be an almost complex structure on W ; we denote also by J the induced automorphism on $W_{\mathbb{C}}$. Consider the following decomposition of $W_{\mathbb{C}}$:

$$\begin{aligned} W^{1,0} &= \{w - iJ(w) \mid w \in W_{\mathbb{R}}\}; \\ W^{0,1} &= \{w + iJ(w) \mid w \in W_{\mathbb{R}}\}. \end{aligned}$$

Of course this decomposition defines a rational Hodge structure on W . Notice that J acts on $W^{1,0}$ as the multiplication by i and on $W^{0,1}$ as the multiplication by $-i$.

Conversely let (W, h) a rational Hodge structure of weight 1 and $W = W^{1,0} \oplus W^{0,1}$ the induced decomposition. We define an automorphism $J : W_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$ setting $J(w) = iw$ if $w \in W^{1,0}$ and $J(w) = -iw$ if $w \in W^{0,1}$. Consider $W_{\mathbb{R}}$ as embedded in $W_{\mathbb{C}}$ via the natural map $W_{\mathbb{R}} \hookrightarrow W_{\mathbb{C}}$ defined by $w \mapsto w \otimes 1$. Then for all $w \in W_{\mathbb{R}}$ there is $v \in W^{1,0}$ such that $w = v + \bar{v}$. We observe that

$$J(w) = J(v + \bar{v}) = iv - i\bar{v} = i(v - \bar{v})$$

is still an element in $W_{\mathbb{R}}$ and thus J defines an almost complex structure on $W_{\mathbb{R}}$. □

Corollary 2.2.14.1. *Let (W, h) be a rational Hodge structure of weight 1 and let J be the corresponding almost complex structure on W . Then $h(a + ib)w = aw + bJ(w)$ for all $a, b \in \mathbb{R}$ and $w \in W$.*

Proof. It follows from Lemma 2.2.14 and the proof of [vG00, Proposition 1.4]. \square

Definition 2.2.15. Let (W, h) be a rational Hodge structure of weight k . An endomorphism $\Phi : W \rightarrow W$ is a Hodge endomorphism if the linear extension $\Phi_{\mathbb{R}} : W_{\mathbb{R}} \rightarrow W_{\mathbb{R}}$

$$\Phi_{\mathbb{R}}(h(z)w) = h(z)(\Phi_{\mathbb{R}}(w)) \quad \forall w \in W_{\mathbb{R}}, z \in \mathbb{C}^*.$$

The induced endomorphism $\Phi_{\mathbb{C}} : W_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$ preserves the Hodge decomposition, i.e. $\Phi(W^{p,q}) \subseteq W^{p,q}$. We denote with $End_{Hod}(W)$ the subalgebra of $End(W)$ of the Hodge endomorphisms.

Proposition 2.2.16. *Let $\Phi \in End_{Hod}(W)$. Then $Ker(\Phi)$ and $Im(\Phi)$ are rational Hodge substructure.*

Definition 2.2.17 ([vG00]). Let (W, h) be a rational Hodge structure of weight k . A polarization on W is a bilinear map:

$$\Psi : W \times W \rightarrow \mathbb{Q}$$

such that, for all $v, w \in W_{\mathbb{R}}$

- $\Psi(h(z)v, h(z)w) = (z\bar{z})^k \Psi(v, w)$;
- $\Psi(v, h(i)w)$ is a symmetric and positive definite form.

Proposition 2.2.18. *Let (X, L) be a polarized complex smooth projective variety where $L \in Pic(X)$ is an ample line bundle on X . Then the cohomology group $H^k(X, \mathbb{Q})$ is a polarized Hodge structure of weight k .*

Proof. See [Voi02, p. 160]. \square

Proposition 2.2.19. *Let G be a finite group and $\rho_i : G \rightarrow GL(V_i)$ its irreducible complex representations. Let (W, h) be a rational Hodge structure of weight 1 and $\tau : G \rightarrow GL(W)$ a rational representation such that $\tau(G) \subset End_{Hod}(W)$. Consider the induced complex representations $\tau_{\mathbb{C}} : G \rightarrow GL(W_{\mathbb{C}})$ and $\rho = \tau|_{W^{1,0}} : G \rightarrow GL(W^{1,0})$. Then:*

- $n_{\tau_{\mathbb{C}}}(\rho_i) = n_{\rho}(\rho_i) + n_{\rho}(\bar{\rho}_i)$;
- if ρ_i is self-dual $n_{\tau_{\mathbb{C}}}(\rho_i)$ is even.

Proof. The subspaces $W^{1,0}$ and $W^{0,1}$ are subrepresentations of $W_{\mathbb{C}}$. Moreover, by Lemma 2.2.14, we deduce that if $\tau_{\mathbb{C}}|_{W^{1,0}} = \rho$ then $\tau_{\mathbb{C}}|_{W^{0,1}} = \bar{\rho}$, i.e. $\tau_{\mathbb{C}} = \rho \oplus \bar{\rho}$. Hence the following equalities hold:

$$\begin{aligned} n_{\tau_{\mathbb{C}}}(\rho_i) &= n_{\rho}(\rho_i) + n_{\bar{\rho}}(\rho_i), \\ n_{\bar{\rho}}(\rho_i) &= n_{\rho}(\bar{\rho}_i). \end{aligned}$$

In particular if ρ_i is self-dual we get $n_{\tau_{\mathbb{C}}}(\rho_i) = 2n_{\rho}(\rho_i)$. \square

Proposition 2.2.20. *Let (W, h) be a rational Hodge structure, G a finite group and let $\tau : G \rightarrow GL(W)$ be a rational representation such that $\tau(G) \subset \text{End}_{\text{Hod}}(W)$. Then the isotypical and isogenous component of τ are Hodge substructures.*

Proof. In the previous section we have defined the isotypical and isogenous components of a given representation $\tau : G \rightarrow GL(W)$ as the images of opportune elements in $\mathbb{Q}[G]$. Moreover $\tau(g) \in \text{End}_{\text{Hod}}(W)$ for all $g \in G$ implies $\tilde{\tau}(x) \in \text{End}_{\text{Hod}}(W)$ for all $x \in \mathbb{Q}[G]$. \square

Remark 2.2.5. A rational polarized weight one Hodge structure defines an isogeny class of Abelian varieties (see [vG00, Section 8]). Moreover this correspondence descends to a correspondence between Hodge substructures and Abelian subvarieties. Under this correspondence the group action decomposition described above corresponds to the decomposition described in [BL04, Section 13.6].

2.3 Group action and Cohomology

We apply the results of the previous section to the cohomology of the curves. We assume the following notation: given a complex representation $\rho : G \rightarrow GL(V)$ of a finite group G we denote by $n_{\rho}(\rho_i)$ the multiplicity of the irreducible complex representation ρ_i in the decomposition of ρ ; given a rational representation $\tau : G \rightarrow GL(W)$ we denote by $n_{\tau}(\tau_j)$ the multiplicity of the irreducible rational representation τ_j in the decomposition of τ .

2.3.1 The Broughton formula

Let C be a smooth curve of genus $g(C)$ and let G be a finite group of automorphisms of C . We will denote by φ the natural action induced by G on the first cohomology group

$$\varphi : G \rightarrow GL(H^1(C, \mathbb{C})) \quad \varphi : g \mapsto (g^{-1})^*.$$

Let assume $C/G \cong \mathbb{P}^1$ and let $T = [g_1, \dots, g_r]$ be the spherical system of generators associated to the ramified covering $f : C \rightarrow C/G \cong \mathbb{P}^1$.

Proposition 2.3.1 ([Bro87]). *Let $\varphi = \oplus_{i=1}^m n_{\varphi}(\rho_i) \rho_i$ be the decomposition of φ into irreducible complex representations. Then, with the notation as above we have:*

- $n_{\varphi}(\rho_1) = \langle \varphi, \rho_1 \rangle = 0,$
- $n_{\varphi}(\rho_i) = \langle \varphi, \rho_i \rangle = \chi_i(1)(r-2) - \sum_{j=1}^r l_{g_j}(\rho_i),$

where χ_i are the characters of the irreducible complex representations $\rho_i : G \rightarrow GL(V_i)$ of G , ρ_1 is the trivial representation, $r = \ell(T)$ is the length of T and $l_{g_j}(\rho_i)$ is the multiplicity of the trivial character in the restriction of ρ_i to $\langle g_j \rangle$.

Proof. Proved in [Bro87, Proposition 2]. □

Remark 2.3.1. The same computations can be done using the Lefschetz fixed point formula (see [GH94, Chapter 3.4]). However, since we are interested only in the first cohomology groups of curves, Broughton's formula makes calculations faster and easier.

The group G induces an action not only on the complex (or real) cohomology, but also in the rational one. These actions are connected since $H^1(C, \mathbb{C}) = H^1(C, \mathbb{Q}) \otimes \mathbb{C}$. We will denote both actions with φ . Applying together Proposition 2.2.3 and Proposition 2.3.1 we can compute the decomposition of φ into irreducible rational representations.

Notice that here we are exactly in the situation described in Proposition 2.2.19: the finite group G acts on $H^1(C, \mathbb{Q})$ that is a rational Hodge structure of weight 1. Moreover, since G acts holomorphically on C , the action on the cohomology preserves the Hodge decomposition and then

$$\varphi(G) \subset \text{End}_{\text{Hod}}(H^1(C, \mathbb{Q})).$$

Example 2.3.1. Let G be the abelian group $\mathbb{Z}_3^2 := (\mathbb{Z}/3\mathbb{Z})^2$. Consider the unmixed ramification structures for G :

$$\begin{aligned} T_1 &= [(1, 1), (2, 1), (1, 1), (1, 2), (1, 1)], \\ T_2 &= [(0, 2), (0, 1), (1, 0), (2, 0)], \end{aligned}$$

of type $([3^5], [3^4])$. We denote by f and h the corresponding ramified coverings of \mathbb{P}^1 :

$$\begin{aligned} f : C &\rightarrow C/G \cong \mathbb{P}^1, \\ h : D &\rightarrow D/G \cong \mathbb{P}^1, \end{aligned}$$

where C and D have genus 7 and 4 respectively.

The character table of G is

	Id	$(1, 0)$	$(2, 0)$	$(0, 1)$	$(0, 2)$	$(1, 1)$	$(2, 2)$	$(2, 1)$	$(1, 2)$
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	ξ_3	ξ_3^2	1	1	ξ_3	ξ_3^2	ξ_3^2	ξ_3
χ_3	1	ξ_3^2	ξ_3	1	1	ξ_3^2	ξ_3	ξ_3	ξ_3^2
χ_4	1	1	1	ξ_3	ξ_3^2	ξ_3	ξ_3^2	ξ_3	ξ_3^2
χ_5	1	ξ_3	ξ_3^2	ξ_3	ξ_3^2	ξ_3^2	ξ_3	1	1
χ_6	1	ξ_3^2	ξ_3	ξ_3	ξ_3^2	1	1	ξ_3^2	ξ_3
χ_7	1	1	1	ξ_3^2	ξ_3	ξ_3^2	ξ_3	ξ_3^2	ξ_3
χ_8	1	ξ_3	ξ_3^2	ξ_3^2	ξ_3	1	1	ξ_3	ξ_3^2
χ_9	1	ξ_3^2	ξ_3	ξ_3	ξ_3	ξ_3	ξ_3^2	1	1

By Proposition 2.2.3 G has 5 irreducible \mathbb{Q} -representations τ_1, \dots, τ_5 with:

$$\begin{aligned}\tau_1 \otimes_{\mathbb{Q}} \mathbb{C} &= \rho_1, \\ \tau_2 \otimes_{\mathbb{Q}} \mathbb{C} &= \rho_2 \oplus \rho_3, \\ \tau_3 \otimes_{\mathbb{Q}} \mathbb{C} &= \rho_4 \oplus \rho_7, \\ \tau_4 \otimes_{\mathbb{Q}} \mathbb{C} &= \rho_5 \oplus \rho_9, \\ \tau_5 \otimes_{\mathbb{Q}} \mathbb{C} &= \rho_6 \oplus \rho_8.\end{aligned}$$

We apply the Broughton formula (Proposition 2.3.1) to compute the decomposition of the action of the group G on $H^1(C, \mathbb{C})$ and $H^1(D, \mathbb{C})$. We get

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9
φ_C	0	3	3	3	1	0	3	0	1
φ_D	0	0	0	0	2	2	0	2	2

for the complex cohomology groups $H^1(C, \mathbb{C})$, $H^1(D, \mathbb{C})$ and

	τ_1	τ_2	τ_3	τ_4	τ_5
φ_C	0	3	3	1	0
φ_D	0	0	0	2	2

for the rational cohomology groups $H^1(C, \mathbb{Q})$, $H^1(D, \mathbb{Q})$.

2.3.2 Cohomology of surfaces isogenous to a product

The cohomology of surfaces isogenous to a product has been studied in [CLZ13, Section 4.5] and in [CL13, Section 1.4]. However the authors focused on the complex cohomology, while we focus on rational cohomology.

Let $S = \frac{C \times D}{G}$ be a surface isogenous to a higher product of unmixed type. Then the second cohomology of S depends on the cohomology of C and D and on the action of G .

First of all we need a topological lemma:

Lemma 2.3.2. *Let $\pi : \tilde{X} \rightarrow X$ be a (topological) covering space of degree N defined by an action of a group G on \tilde{X} . Then with coefficients in a field F whose characteristic is 0 or a prime not dividing n , the map $\pi^* : H^k(X, F) \rightarrow H^k(\tilde{X}, F)$ is injective with image the subgroup $H^k(\tilde{X}, F)^G$.*

Proof. Proved in [Hat02, Proposition 3G.1]. \square

Remark 2.3.2. In the lemma above with (topological) covering we mean unramified covering.

Proposition 2.3.3. *Let $S = \frac{C \times D}{G}$ be a surface isogenous to a higher product of unmixed type. Then the second cohomology group of S is given by $H^2(S, \mathbb{Q}) \cong U \oplus Z$, where*

$$\begin{aligned} U &:= (H^2(C, \mathbb{Q}) \otimes H^0(D, \mathbb{Q})) \oplus (H^0(C, \mathbb{Q}) \otimes H^2(D, \mathbb{Q})), \\ Z &:= (H^1(C, \mathbb{Q}) \otimes H^1(D, \mathbb{Q}))^G. \end{aligned}$$

Proof. We compute the second cohomology of $C \times D$ with the Künneth formula (see [Hat02, Theorem 3.16]) and we apply Lemma 2.3.2. Since G acts trivially on the zero cohomology and on the second cohomology of the curves C and D we get the result. \square

Remark 2.3.3. Consider $H^2(S, \mathbb{Q})$ as polarized Hodge structure of weight 2. Then $U, Z \leq H^2(S, \mathbb{Q})$ are Hodge substructures.

The subspace U has dimension 2, and $U \otimes_{\mathbb{Q}} \mathbb{C} \leq H^{1,1}(S)$. Then, as rational Hodge structure, it is isomorphic to the Tate structure $\mathbb{Q}^2(-1)$. It follows that $H^2(S, \mathbb{Q})$ is determined, as Hodge structure, by Z .

Lemma 2.3.4. *Let G be a finite group and let $\rho_i : G \rightarrow GL(V_i)$ $i = 1, \dots, m$ be its irreducible complex representations, where ρ_1 is the trivial representation. Then*

$$n_{\rho_i \otimes \rho_j}(\rho_1) = \langle \rho_i \otimes \rho_j, \rho_1 \rangle = \begin{cases} 1 & \text{if } \rho_j = \overline{\rho_i}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. See [CLZ13, Section 4.5] \square

Proposition 2.3.5. *Let G be a finite group and let $\tau_j : G \rightarrow GL(W_j)$ $j = 1, \dots, t$ be its irreducible rational representations, where τ_1 is the trivial representation. Then the multiplicity of the trivial representation in $\tau_j \otimes \tau_k$ is:*

$$n_{\tau_j \otimes \tau_k}(\tau_1) = \begin{cases} s^2[K_i : \mathbb{Q}] & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases}$$

where $\tau_j \otimes \mathbb{C} = s \bigoplus_{\sigma \in \text{Gal}(K_i/\mathbb{Q})} \sigma(\rho_i)$.

Proof. It follows from Proposition 2.3.4 and Proposition 2.2.3. \square

Let $\tau_{j_1} : G \rightarrow GL(W_1)$ and $\tau_{j_2} : G \rightarrow GL(W_2)$ be two irreducible rational representations of G . The group acts trivially on $(W_{j_1} \otimes W_{j_2})^G$ and then

$$\dim(W_{j_1} \otimes W_{j_2})^G = n_{\tau_{j_1} \otimes \tau_{j_2}}(\tau_1).$$

In particular $\dim(W_{j_1} \otimes W_{j_2})^G \neq 0$ if and only if $j_1 = j_2$, and in this case the dimension is determined by Proposition 2.3.5.

Let $S = \frac{C \times D}{G}$ be a surface isogenous to a higher product of unmixed type and let $\tau_j : G \rightarrow GL(W_j)$, $j = 1, \dots, t$ be the irreducible rational representations of G . Let $\varphi_C : G \rightarrow GL(H^1(C, \mathbb{Q}))$ and $\varphi_D : G \rightarrow GL(H^1(D, \mathbb{Q}))$ be the actions induced by G on the first cohomology of curves:

$$\begin{aligned} \varphi_C &= n_C(\tau_1)\tau_1 \oplus \dots \oplus n_C(\tau_t)\tau_t, \\ \varphi_D &= n_D(\tau_1)\tau_1 \oplus \dots \oplus n_D(\tau_t)\tau_t. \end{aligned}$$

Lemma 2.3.6. *The Hodge substructure Z decomposes as*

$$Z \cong \bigoplus_{j=1}^t n_C(\tau_j) n_D(\tau_j) (W_j \otimes W_j)^G \quad (2.10)$$

Proof. Given two irreducible rational representations $\tau_{j_1} : G \rightarrow GL(W_{j_1})$ and $\tau_{j_2} : G \rightarrow GL(W_{j_2})$ of G they define a Hodge substructure of Z isomorphic to

$$n_C(\tau_{j_1}) n_D(\tau_{j_2}) (W_{j_1} \otimes W_{j_2})^G.$$

We observed that $\dim(W_{j_1} \otimes W_{j_2})^G \neq 0$ if and only if $j_1 = j_2$. □

From now on we focus on the case of surfaces isogenous to a higher product with $\chi(\mathcal{O}_S) = 2$ and $q(S) = 0$. How does the decomposition (2.10) work for this kind of surfaces?

Proposition 2.3.7. *Let $S = \frac{C \times D}{G}$ be a surface isogenous to a higher product with $\chi(\mathcal{O}_S) = 2$ and $q(S) = 0$. Then one of the following cases holds:*

- a) *There exists an absolutely irreducible rational representation $\tau : G \rightarrow GL(W)$ such that*

$$\begin{aligned} n_C(\tau) &= n_D(\tau) = 2, \\ n_C(\tau_j) \cdot n_D(\tau_j) &= 0, \quad \forall \tau_j \text{ different from } \tau. \end{aligned}$$

- b) *There exists an irreducible rational representation $\tau : G \rightarrow GL(W)$ and an irreducible complex representation $\rho : G \rightarrow GL(V)$ with $\tau_{\mathbb{C}} = 2\rho$ such that*

$$\begin{aligned} n_C(\tau) &= n_D(\tau) = 1, \\ n_C(\tau_j) \cdot n_D(\tau_j) &= 0, \quad \forall \tau_j \text{ different from } \tau. \end{aligned}$$

c) *There exists an irreducible rational representation $\tau : G \rightarrow GL(W)$ and an irreducible complex representation $\rho : G \rightarrow GL(V)$ with $\tau_{\mathbb{C}} = \rho \oplus \bar{\rho}$ such that*

$$\begin{aligned} n_C(\tau) &= 1, \quad n_D(\tau) = 2, \\ n_C(\tau_j) \cdot n_D(\tau_j) &= 0, \quad \forall \tau_j \text{ different from } \tau. \end{aligned}$$

d) *There exist two irreducible rational representations $\tau_{j_1} : G \rightarrow GL(W_{j_1})$, $\tau_{j_2} : G \rightarrow GL(W_{j_2})$ and two irreducible complex representations $\rho_{i_1} : G \rightarrow GL(V_{i_1})$, $\rho_{i_2} : G \rightarrow GL(V_{i_2})$ with $\tau_{j_1} \otimes \mathbb{C} = \rho_{i_1} \oplus \bar{\rho}_{i_1}$, $\tau_{j_2} \otimes \mathbb{C} = \rho_{i_2} \oplus \bar{\rho}_{i_2}$ and $j_1 \neq j_2$ such that*

$$\begin{aligned} n_C(\tau_{j_1}) &= n_C(\tau_{j_2}) = n_D(\tau_{j_1}) = n_D(\tau_{j_2}) = 1, \\ n_C(\tau_j) \cdot n_D(\tau_j) &= 0, \quad \forall \tau_j \text{ different from } \tau_{j_1}, \tau_{j_2}. \end{aligned}$$

Proof. For a surface S isogenous to a higher product with $\chi(\mathcal{O}_S) = 2$ and $q(S) = 0$ we have $\dim_{\mathbb{Q}} Z = 4$.

In the decomposition (2.10) in the Lemma above consider first the case that only one τ_j occurs, so: $Z = n_j m_j Z_j$ where $Z_j = (W_j \otimes W_j)^G$. Since the case $\dim Z_j = 1$, $n_C(\tau_j) = 4$, $n_D(\tau_j) = 1$ is excluded by Proposition 2.2.19 we have, up to interchange C and D , three possibilities:

- $\dim Z_j = 1$, $n_C(\tau_j) = n_D(\tau_j) = 2$: this is the case a ;
- $\dim Z_j = 2$, $n_C(\tau_j) = 2$ and $n_D(\tau_j) = 1$: this is the case c ;
- $\dim Z_j = 4$, $n_C(\tau_j) = n_D(\tau_j) = 1$: this is the case b .

There is only one more option, when two τ_j 's occur in (2.10). Thus $Z = Z_{j_1} \oplus Z_{j_2}$ with $\dim Z_{j_1} = \dim Z_{j_2} = 2$. This is the case d . \square

As already mentioned, surfaces isogenous to a higher product of unmixed type with $\chi(\mathcal{O}_S) = 2$ and $q(S) = 0$ have been classified by Gleissner. In [Gle13] he proves that only 21 groups admit an unmixed ramification structure such that the corresponding surface has $\chi(\mathcal{O}_S) = 2$ and $q(S) = 0$. In particular 7 groups admit more than one non-isomorphic structures, and he obtains 32 families of surfaces isogenous to a higher product of unmixed type with $\chi(\mathcal{O}_S) = 2$ and $q(S) = 0$. A complete list can be found in Table 2.1. The explicit forms of the unmixed ramification structures can be found in [Gle13] For all the surfaces in the list we studied the Hodge substructure Z , according to Proposition 2.3.7.

Let G be one of the 14 groups in the following list:

$$\begin{aligned} &(\mathbb{Z}_2)^3 \rtimes_{\varphi} \mathcal{S}_4, \quad (\mathbb{Z}_2)^4 \rtimes_{\varphi} \mathcal{D}_5, \quad \mathcal{S}_5, \quad (\mathbb{Z}_2)^4 \rtimes_{\psi} \mathcal{D}_3, \\ &U(4, 2), \quad \mathcal{A}_5, \quad \mathcal{S}_4 \times \mathbb{Z}_2, \quad \mathcal{D}_4 \times (\mathbb{Z}_2)^2, \quad (\mathbb{Z}_2)^4 \rtimes_{\varphi} \mathbb{Z}_2, \\ &\mathcal{S}_4, \quad \mathcal{D}_4 \times \mathbb{Z}_2, \quad (\mathbb{Z}_2)^2 \rtimes_{\varphi} \mathbb{Z}_4, \quad (\mathbb{Z}_2)^4, \quad (\mathbb{Z}_2)^3. \end{aligned}$$

For all the irreducible complex representations $\rho : G \rightarrow GL(V)$ we get $K_\rho \subseteq \mathbb{R}$ and the Schur index of ρ is equal 1. Therefore a surface S with a such group is forced to be of type a .

We verified that also the surfaces related to the groups

$$PSL(2, \mathbb{F}_7) \times \mathbb{Z}_2, \quad PSL(2, \mathbb{F}_7), \quad (\mathbb{Z}_2)^3 \rtimes_{\varphi} \mathcal{D}_4$$

are of type a , although these groups admit irreducible complex representations with $K_\rho \not\subseteq \mathbb{R}$.

Example 2.3.2. As example we study in detail the group $G = PSL(2, \mathbb{F}_7)$. G has 6 irreducible complex representations ρ_1, \dots, ρ_6 associated to the charaters χ_1, \dots, χ_6 :

	Id	2	3	4	$7a$	$7b$
χ_1	1	1	1	1	1	1
χ_2	3	-1	0	1	ξ	$\bar{\xi}$
χ_3	3	-1	0	1	$\bar{\xi}$	ξ
χ_4	6	2	0	0	-1	-1
χ_5	7	-1	1	-1	0	0
χ_6	8	0	-1	0	1	1

where $\xi = \frac{-1+i\sqrt{7}}{2}$. By Proposition 2.2.3, G has only 5 irreducible rational representations τ_1, \dots, τ_5 . One has

$$\begin{aligned} \tau_1 \otimes_{\mathbb{Q}} \mathbb{C} &= \rho_1, \\ \tau_2 \otimes_{\mathbb{Q}} \mathbb{C} &= \rho_2 \oplus \rho_3, \\ \tau_3 \otimes_{\mathbb{Q}} \mathbb{C} &= \rho_4, \\ \tau_4 \otimes_{\mathbb{Q}} \mathbb{C} &= \rho_5, \\ \tau_5 \otimes_{\mathbb{Q}} \mathbb{C} &= \rho_6. \end{aligned}$$

The group $PSL(2, \mathbb{F}_7)$ admits two non-isomorphic unmixed structures (T_{C_1}, T_{D_1}) and (T_{C_2}, T_{D_2}) of types $([7^3], [3^2, 4])$ and $([3^2, 7], [4^3])$ respectively. Since there is only one conjugacy class of elements of order 3 and one conjugacy class of elements of order 4 in G (denoted in the table above with 3 and 4), we can apply the Broughton formula to the curves D_1 and D_2 easily. We get:

	τ_1	τ_2	τ_3	τ_4	τ_5
φ_{D_1}	0	0	0	0	2
φ_{D_2}	0	0	0	4	2

So, even if G has two non self-dual representations (ρ_2 and ρ_3), the surfaces isogenous to a higher product associated to both (T_{C_1}, T_{D_1}) and (T_{C_2}, T_{D_2}) are of type a .

Finally surfaces related to the groups

$$G(128, 36), \quad (\mathbb{Z}_2)^4 \rtimes_{\varphi} \mathcal{D}_3, \quad (\mathbb{Z}_2)^3 \rtimes_{\varphi} \mathbb{Z}_4, \quad (\mathbb{Z}_3)^2,$$

are not of type a and we will study them in the next section. The complete list, with the corresponding type, is summarized in Table 2.1, at the end of this section.

Theorem 2.3.8. *Let $S = \frac{C \times D}{G}$ be a surface isogenous to a higher product of unmixed type with $\chi(\mathcal{O}_S) = 2$, $q(S) = 0$ and assume that S is of type a . Then there exist two elliptic curves E_C and E_D such that $H^2(S, \mathbb{Q}) \cong H^2(E_C \times E_D, \mathbb{Q})$ as rational Hodge structures.*

Proof. The proof consists of two steps: in the first one we construct the two elliptic curves E_C and E_D ; in the second one we prove that $H^2(S, \mathbb{Q}) \cong H^2(E_C \times E_D, \mathbb{Q})$.

Step 1: By hypothesis there exists an absolutely irreducible rational representation $\tau : G \rightarrow GL(W)$ such that $n_C(\tau) = n_D(\tau) = 2$; let $\dim W = n$. We denote by A_C and A_D the isotypical components related to τ in $H^1(C, \mathbb{Q})$ and $H^1(D, \mathbb{Q})$: A_C and A_D are, at the same time, rational Hodge substructure and G -subrepresentations of dimension $2n$ and we obtain

$$Z \cong (A_C \otimes A_D)^G,$$

where Z is the Hodge substructure defined in Proposition 2.3.3. We consider now the isogenous decomposition of A_C and A_D . Since τ is an absolutely irreducible rational representation, the corresponding skew-field \mathbb{D} is simply \mathbb{Q} . So, by Proposition 2.2.10, we get $A_C \cong B_C^{\oplus n}$ and $A_D \cong B_D^{\oplus n}$ where B_C and B_D are Hodge substructures, but no longer G -subrepresentations, of A_C and A_D with $\dim B_C = \dim B_D = 2$. Now, by Remark 2.2.5, there exists two elliptic curves E_C and E_D , defined up to isogeny, such that

$$B_C \cong H^1(E_C, \mathbb{Q}), \quad B_D \cong H^1(E_D, \mathbb{Q}),$$

as rational Hodge structures.

Step 2: The Hodge structures of weight two Z and $B_C \otimes B_D$ have the same dimension and the same Hodge numbers; in particular $\dim Z^{2,0} = \dim(B_C \otimes B_D)^{2,0} = 1$. The action of G provides a Hodge homomorphism $A_C \otimes A_D \rightarrow Z$: by restriction we get a map $\psi : B_C \otimes B_D \rightarrow Z$. Consider the Hodge substructure $\text{Im}(\psi)$. We can assume that $\dim \text{Im}(\psi)^{2,0} = 1$: otherwise we have to change the choice of B_C and B_D in A_C and A_D . If ψ is an isomorphism we are done. Otherwise let $k := \dim \text{Ker}(\psi)$. We have the decompositions:

$$B_C \otimes B_D \simeq P \oplus \text{Ker}(\psi), \quad Z \simeq \text{Im}(\psi) \oplus \mathbb{Q}^k(-1),$$

where P is a Hodge substructure with $\dim P^{2,0} = 1$ and $\dim P^{1,1} = 2 - k$. The Hodge structures $B_C \otimes B_D$ and Z are isomorphic since

- ψ defines an isomorphism between P and $\text{Im}(\psi)$,
- $\dim \text{ker}(\psi)^{2,0} = 0$ and then $\text{ker}(\psi) \simeq \mathbb{Q}^k(-1)$.

□

G	$ G $	SGL	$g(C)$	$g(D)$	type
$PSL(2, \mathbb{F}_7) \times \mathbb{Z}_2$	336	$\langle 336, 209 \rangle$	17	43	a
$(\mathbb{Z}_2)^3 \rtimes_{\varphi} \mathcal{S}_4$	192	$\langle 192, 995 \rangle$	49	9	a
$PSL(2, \mathbb{F}_7)$	168	$\langle 168, 42 \rangle$	49	8	a
$PSL(2, \mathbb{F}_7)$	168	$\langle 168, 42 \rangle$	17	22	a
$(\mathbb{Z}_2)^4 \rtimes_{\varphi} \mathcal{D}_5$	160	$\langle 160, 234 \rangle$	5	81	a
$G(128, 36)$	128	$\langle 128, 36 \rangle$	17	17	b
\mathcal{S}_5	120	$\langle 120, 34 \rangle$	9	31	a
$(\mathbb{Z}_2)^4 \rtimes_{\varphi} \mathcal{D}_3$	96	$\langle 96, 195 \rangle$	5	49	c
$(\mathbb{Z}_2)^4 \rtimes_{\psi} \mathcal{D}_3$	96	$\langle 96, 227 \rangle$	25	9	a
$(\mathbb{Z}_2)^3 \rtimes_{\varphi} \mathcal{D}_4$	64	$\langle 64, 73 \rangle$	9	17	a
$U(4, 2)$	64	$\langle 64, 138 \rangle$	9	17	a
\mathcal{A}_5	60	$\langle 60, 5 \rangle$	13	11	a
\mathcal{A}_5	60	$\langle 60, 5 \rangle$	41	4	a
\mathcal{A}_5	60	$\langle 60, 5 \rangle$	9	16	a
\mathcal{A}_5	60	$\langle 60, 5 \rangle$	5	31	a
$\mathcal{S}_4 \times \mathbb{Z}_2$	48	$\langle 48, 48 \rangle$	5	25	a
$\mathcal{S}_4 \times \mathbb{Z}_2$	48	$\langle 48, 48 \rangle$	9	13	a
$\mathcal{S}_4 \times \mathbb{Z}_2$	48	$\langle 48, 48 \rangle$	13	9	a
$\mathcal{S}_4 \times \mathbb{Z}_2$	48	$\langle 48, 48 \rangle$	3	49	a
$(\mathbb{Z}_2)^3 \rtimes_{\varphi} \mathbb{Z}_4$	32	$\langle 32, 22 \rangle$	9	9	d
$\mathcal{D}_4 \times (\mathbb{Z}_2)^2$	32	$\langle 32, 46 \rangle$	9	9	a
$(\mathbb{Z}_2)^4 \rtimes_{\varphi} \mathbb{Z}_2$	32	$\langle 32, 27 \rangle$	17	5	a
$(\mathbb{Z}_2)^4 \rtimes_{\varphi} \mathbb{Z}_2$	32	$\langle 32, 27 \rangle$	9	9	a
\mathcal{S}_4	24	$\langle 24, 12 \rangle$	5	13	a
\mathcal{S}_4	24	$\langle 24, 12 \rangle$	3	25	a
$\mathcal{D}_4 \times \mathbb{Z}_2$	16	$\langle 16, 11 \rangle$	9	5	a
$(\mathbb{Z}_2)^2 \rtimes_{\varphi} \mathbb{Z}_4$	16	$\langle 16, 3 \rangle$	9	5	a
$(\mathbb{Z}_2)^4$	16	$\langle 16, 14 \rangle$	9	5	a
$\mathcal{D}_4 \times \mathbb{Z}_2$	16	$\langle 16, 11 \rangle$	3	17	a
$(\mathbb{Z}_3)^2$	9	$\langle 9, 2 \rangle$	7	4	c
$(\mathbb{Z}_2)^3$	8	$\langle 8, 5 \rangle$	5	5	a
$(\mathbb{Z}_2)^3$	8	$\langle 8, 5 \rangle$	3	9	a

Table 2.1: Complete list of groups that admit an unmixed ramification structure such that the corresponding surfaces S isogenous to a higher product has $\chi(\mathcal{O}_S) = 2$ and $q(S) = 0$. SGL is the pair that identifies the group in the Small Groups Library (on Magma).

2.4 The exceptional cases

In this section we study one by one the families of surfaces in Table 2.1 not of type a .

Given a finite group G and an unmixed ramification structure (T_C, T_D) for G we will use the following notation:

- $f : C \rightarrow \mathbb{P}^1$ and $h : D \rightarrow \mathbb{P}^1$ are the Galois covering associated to the spherical system of generators T_C and T_D ;
- $S = \frac{C \times D}{G}$ is the surface isogenous to a product of unmixed type corresponding to the unmixed ramification structure;
- $Z \subset H^2(S, \mathbb{Q})$ is the 4-dimensional Hodge substructure of type $(1, 2, 1)$ defined by

$$Z = (H^1(C, \mathbb{Q}) \otimes H^1(D, \mathbb{Q}))^G.$$

Let $\tau : G \rightarrow GL(W)$ be an irreducible rational representation of G and let A_C, A_D be the isotypical components of $H^1(C, \mathbb{Q})$ and $H^1(D, \mathbb{Q})$ related to τ . Assume that $Z \cong (A_C \otimes A_D)^G$: as described in the proof of Theorem 2.3.7 this is exactly what happens for surfaces of type a, b and c .

Let $H \triangleleft G$ be the normal subgroup $H = \ker(\tau)$. Then we get

$$\begin{aligned} Z &= (H^1(C, \mathbb{Q}) \otimes H^1(D, \mathbb{Q}))^G = \\ &= (H^1(C, \mathbb{Q})^H \otimes H^1(D, \mathbb{Q})^H)^{G/H}. \end{aligned} \quad (2.11)$$

Remark 2.4.1. Notice that, for a general subgroup $H \leq G$, we have

$$(H^1(C, \mathbb{Q}) \otimes H^1(D, \mathbb{Q}))^H \not\cong (H^1(C, \mathbb{Q})^H \otimes H^1(D, \mathbb{Q})^H).$$

For example for $H = G$ we get the Hodge structure Z in the first case and the empty vector space in second one, since $C/G \cong D/G \cong \mathbb{P}^1$. The equation (2.11) holds because our specific choice of the subgroup H .

Using this idea (with appropriate modifications for the case d) we extend the result of Theorem 2.3.8 to the remaining cases.

2.4.1 Case b

The only case is the finite group $G = G(128, 36)$ with presentation:

$$G = \left\langle g_1, \dots, g_7 \mid \begin{array}{lll} g_1^2 = g_4 & g_2^2 = g_5 & g_2^{g_1} = g_2 g_3 \\ g_3^{g_1} = g_3 g_6 & g_3^{g_2} = g_3 g_7 & g_4^{g_2} = g_4 g_6 \\ g_5^{g_1} = g_5 g_7 & & \end{array} \right\rangle,$$

where $g_i^{g_j} := g_j^{-1} g_i g_j$; G has order 128 and it is determined by the pair $\langle 128, 36 \rangle$ in the Small Groups Library on Magma.

Consider the unmixed ramification structure (T_C, T_D) of type $([4^3], [4^3])$:

$$T_C = [g_1 g_2 g_4 g_6, g_1 g_4 g_5 g_6, g_2 g_3 g_4 g_7], \quad T_D = [g_1 g_2 g_3 g_6 g_7, g_2 g_5 g_7, g_1 g_3 g_4 g_7].$$

By direct computation (using [Magma]) we verify that the corresponding surface isogenous to a product S is of type b , i.e. there exists an irreducible rational representation $\tau : G \rightarrow GL(W)$, $\dim W = 4$ and an irreducible complex representation $\rho : G \rightarrow GL(V)$, $\dim V = 2$ with $\tau_{\mathbb{C}} = 2\rho$ such that

$$\begin{aligned} n_C(\tau) &= n_D(\tau) = 1, \\ n_C(\tau_j) \cdot n_D(\tau_j) &= 0 \quad \forall \tau_j \text{ different from } \tau. \end{aligned}$$

Let $H \triangleleft G$ be the normal subgroup $H := \text{Ker}(\tau)$: a set of generators for H is

$$H = \langle g_7, g_6, g_3g_4, g_4g_5 \rangle.$$

The quotient group G/H has order 8 and it is isomorphic to the quaternion group Q_8 , studied in Example 2.2.2. Consider the intermediate coverings:

$$\begin{array}{ccc} C & \xrightarrow{H} & C' \\ & \searrow G & \downarrow Q_8 \\ & & \mathbb{P}^1 \end{array} \quad \begin{array}{ccc} D & \xrightarrow{H} & D' \\ & \searrow G & \downarrow Q_8 \\ & & \mathbb{P}^1 \end{array}$$

The curves C' and D' have genus 2, by Riemann-Hurwitz formula, and Q_8 acts on their rational cohomology by the rational representation of dimension 4 described in Example 2.2.2.

By construction, since we took $H = \ker(\tau)$, we obtain

$$Z = (H^1(C, \mathbb{Q}) \otimes H^1(D, \mathbb{Q}))^G = (H^1(C, \mathbb{Q})^H \otimes H^1(D, \mathbb{Q})^H)^{Q_8}.$$

In other words we get

$$H^2(S, \mathbb{Q}) \cong H^2(C' \times D', \mathbb{Q})^{Q_8}.$$

Remark 2.4.2. Notice that the quotient surface $\frac{C' \times D'}{Q_8}$ could be singular. For this reason we write $H^2(C' \times D', \mathbb{Q})^{Q_8}$ instead of $H^2\left(\frac{C' \times D'}{Q_8}, \mathbb{Q}\right)$.

The curves C' and D' have genus 2 and they admit the quaternion group Q_8 as automorphism group. Then C' and D' are uniquely determined and they are isomorphic (see Appendix A.2 for a complete proof).

Proposition 2.4.1. *Let S be the surface isogenous to a higher product defined above. Then $H^2(S, \mathbb{Q}) \cong H^2(E_{\sqrt{-2}} \times E_{\sqrt{-2}}, \mathbb{Q})$ where $E_{\sqrt{-2}}$ is the elliptic curve*

$$E_{\sqrt{-2}} = \frac{\mathbb{C}}{\mathbb{Z} \oplus \sqrt{-2}\mathbb{Z}}.$$

Proof. By Corollary A.2.3.1 we get

$$\text{Jac}(C') \sim \text{Jac}(D') \sim E_{\sqrt{-2}} \times E_{\sqrt{-2}}$$

Then the action of Q_8 induces a Hodge morphism ψ

$$\psi : H^1(E_{\sqrt{-2}}, \mathbb{Q}) \otimes H^1(E_{\sqrt{-2}}, \mathbb{Q}) \rightarrow Z.$$

Now, arguing as in the proof of the Theorem 2.3.8, we conclude that

$$H^2(S, \mathbb{Q}) \cong H^2(C' \times D', \mathbb{Q})^{Q_8} \cong H^2(E_{\sqrt{-2}} \times E_{\sqrt{-2}}, \mathbb{Q}).$$

□

Corollary 2.4.1.1. *In particular the Picard number of the surface S coincides with the Picard number of the Abelian surface $E_{\sqrt{-2}} \times E_{\sqrt{-2}}$ and so we get $\rho(S) = 4$. Since $h^{1,1}(S) = 4$, S is a surface with maximal Picard number.*

Remark 2.4.3. The covering maps $f : C \rightarrow \mathbb{P}^1$ and $h : D \rightarrow \mathbb{P}^1$ have both 3 branching values. Then:

- The curves C and D are determined up to isomorphism, by the Riemann Existence Theorem;
- The pair (C, f) and (D, g) are Belyi pairs and then they can be studied with the theory of the dessin d'enfants;
- The surface S is a Beauville surface by Lemma 2.1.18.

2.4.2 Case c

In this case two groups occur. Let G be the finite group $(\mathbb{Z}_3)^2$ and consider the unmixed ramification structure (T_C, T_D) :

$$\begin{aligned} T_C &= [(1, 1), (2, 1), (1, 1), (1, 2), (1, 1)]; \\ T_D &= [(0, 2), (0, 1), (1, 0), (2, 0)]. \end{aligned}$$

This structure has been already studied in Example 2.3.1: notice that the corresponding surface isogenous to a product S is of type c .

In particular, using the notation of Example 2.3.1, there is an irreducible rational representation $\tau_4 : G \rightarrow GL(W_4)$ such that

- $\tau_4 \otimes \mathbb{C} = \rho_5 \oplus \rho_9$;
- $n_C(\tau_4) = 1$ and $n_D(\tau_4) = 2$.

Let H be the normal subgroup $H := \text{Ker}(\rho_5) = \text{Ker}(\rho_9)$: a set of generators for H is

$$H = \langle (2, 1) \rangle.$$

Notice that $H \cong \mathbb{Z}_3$ and also $G/H \cong \mathbb{Z}_3$. Let us consider the intermediate coverings $C' = C/H$ and $D' = D/H$. By Lemma 2.1.8 we have $g(C') = 1$

and $g(D') = 2$.

The curve D' is a curve of genus 2 with an automorphism σ of order 3 such that $D'/\langle\sigma\rangle \simeq \mathbb{P}^1$. It follows that the Jacobian $Jac(D')$ is isogenous to a selfproduct of elliptic curves: $Jac(D') \sim E_D \times E_D$, as proved in Appendix A.1.

Proposition 2.4.2. *Let S be the surface isogenous to a product of unmixed type associated to the unmixed structure (T_C, T_D) . Then $H^2(S, \mathbb{Q}) \simeq H^2(C' \times E_D, \mathbb{Q})$, where C' and E_D are the elliptic curves described above.*

Proof. By construction we have $H^2(S, \mathbb{Q}) \simeq H^2(C' \times D', \mathbb{Q})^G$ and we have already noticed that the cohomology group $H^1(D', \mathbb{Q})$ decomposes as sum of two Hodge substructures, both of dimension 2. Now we conclude with the same arguments used in the proof of Theorem 2.3.8. \square

The case of the group $G = (\mathbb{Z}_2)^4 \rtimes_{\varphi} \mathcal{D}_3$ follows in a similar way. This group has 14 irreducible complex representations with Schur index 1: 12 are self-dual while the remaining two are in the same Galois-orbit. So we have an irreducible rational representation $\tau : G \rightarrow GL(W)$ such that $\tau \otimes \mathbb{C}$ decompose as sum of two irreducible complex representations. We set $H = Ker(\tau)$ and we proceed as before.

2.4.3 Case d

The only group is $G = (\mathbb{Z}_2)^3 \rtimes_{\varphi} \mathbb{Z}_4$ where $\varphi : \mathbb{Z}_4 \rightarrow Aut(\mathbb{Z}_2^3) \simeq GL(3, \mathbb{F}_2)$ is defined by

$$\varphi(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Consider the unmixed ramification structure (T_C, T_D) of type $([2^2, 4^2], [2^2, 4^2])$ for G :

$$\begin{aligned} T_C &= [((1, 0, 0), 2), ((1, 1, 1), 2), ((0, 1, 0), 1), ((0, 0, 1), 3)], \\ T_D &= [((1, 1, 0), 0), ((1, 0, 0), 0), ((1, 0, 0), 3), ((1, 1, 1), 1)]. \end{aligned}$$

We construct the group G in [Magma]:

```
H:=CyclicGroup(4);
K:=SmallGroup(8,5);
A:=AutomorphismGroup(K);
M:=hom<K->K| [K.1->K.1*K.3, K.2->K.2, K.3->K.3]>;
Phi:=hom<H->A| [H.1->M]>;
G,a,b:=SemidirectProduct(K,H,Phi);
G1:=a(K.1);
G2:=a(K.2);
G3:=a(K.3);
G4:=b(H.1);
```

With this notation the unmixed ramification structure is given by:

$$T_C = [g_1 g_4^2, g_1 g_2 g_3 g_4^2, g_2 g_4, g_3 g_4^3], \quad T_D = [g_1 g_2, g_1, g_1 g_4^3, g_1 g_2 g_3 g_4].$$

By direct calculation we see that the surface S is of type d . We denote by $\tau_{j_1} : G \rightarrow GL(W_{j_1})$, $\tau_{j_2} : G \rightarrow GL(W_{j_2})$ the two irreducible rational representations such that

$$\begin{aligned} n_C(\tau_{j_1}) &= n_C(\tau_{j_2}) = n_D(\tau_{j_1}) = n_D(\tau_{j_2}) = 1, \\ n_C(\tau_j) \cdot n_D(\tau_j) &= 0, \quad \forall j \text{ different from } j_1, j_2. \end{aligned}$$

We set $H_1 := \ker(\tau_{j_1})$ and $H_2 := \ker(\tau_{j_2})$ of G . A set of generators for H_1 and H_2 are

$$\begin{aligned} H_1 &= \langle ((1, 0, 0), 0), ((0, 0, 1), 0), ((0, 1, 0), 2) \rangle = \langle g_1, g_3, g_2 g_4^2 \rangle, \\ H_2 &= \langle ((1, 1, 0), 0), ((0, 0, 1), 0), ((0, 1, 0), 2) \rangle = \langle g_1 g_2, g_3, g_2 g_4^2 \rangle. \end{aligned}$$

We observe that:

- $G/H_1 \cong G/H_2 \cong \mathbb{Z}_4$;
- the curves $C_1 := C/H_1$, $C_2 := C/H_2$, $D_1 := D/H_1$ and $D_2 := D/H_2$ have genus 1.

Consider the intermediate coverings:

$$\begin{array}{ccc} C & \xrightarrow{H_i} & C_i \\ & \searrow G & \downarrow \mathbb{Z}_4 \\ & & \mathbb{P}^1 \end{array} \quad \begin{array}{ccc} D & \xrightarrow{H_i} & D_i \\ & \searrow G & \downarrow \mathbb{Z}_4 \\ & & \mathbb{P}^1 \end{array}$$

Since C_i and D_i , $i = 1, 2$ are elliptic curves with an automorphism of order 4 they are all isogenous to

$$E_i = \frac{\mathbb{C}}{\mathbb{Z} \oplus i\mathbb{Z}}.$$

We get

$$Z = (H^1(C_1, \mathbb{Q}) \otimes H^1(D_1, \mathbb{Q}))^G \oplus (H^1(C_2, \mathbb{Q}) \otimes H^1(D_2, \mathbb{Q}))^G.$$

Proposition 2.4.3. *Let S be the surface isogenous to a higher product defined above. Then $H^2(S, \mathbb{Q}) = H^2(E_i \times E_i, \mathbb{Q})$, as rational Hodge structures.*

Proof. We have already observed that

$$Z = (H^1(C_1, \mathbb{Q}) \otimes H^1(D_1, \mathbb{Q}))^G \oplus (H^1(C_2, \mathbb{Q}) \otimes H^1(D_2, \mathbb{Q}))^G.$$

Up to exchange of $C_1 \times D_1$ with $C_2 \times D_2$, we can assume that the Hodge structure $W := (H^1(C_1, \mathbb{Q}) \otimes H^1(D_1, \mathbb{Q}))^G$ has dimension 2 and $\dim W^{2,0} = \dim W^{0,2} = 1$. Now following the same idea of the proof of Theorem 2.3.8 we get:

$$H^2(S, \mathbb{Q}) \cong H^2(C_1 \times D_1, \mathbb{Q}) \cong H^2(E_i \times E_i, \mathbb{Q}).$$

□

Corollary 2.4.3.1. *Consider, as in the proof of Theorem 2.3.8, the Hodge morphism $\psi : H^1(C_1, \mathbb{Q}) \otimes H^1(D_1, \mathbb{Q}) \rightarrow Z$. Here it is clear that ψ is not an isomorphism since its image $\text{Im}(\psi)$ has dimension 2.*

Corollary 2.4.3.2. *For the same reasons of Corollary 2.4.1.1, S is a surface with maximal Picard number.*

2.5 Conclusion

Theorem 2.5.1. *Let S be a surface isogenous to a higher product of unmixed type with $\chi(\mathcal{O}_S) = 2$ and $q(S) = 0$. Then there exist two elliptic curves E_1 and E_2 such that $H^2(S, \mathbb{Q}) \simeq H^2(E_1 \times E_2, \mathbb{Q})$ as rational Hodge structures.*

Proof. It follows from Theorem 2.3.8 and the analysis, case by case, of the previous section. \square

Remark 2.5.1. This result does not come from the existence of an intermediate covering, i.e. there do not exist morphisms φ and ψ such that the diagram

$$\begin{array}{ccc} C \times D & \xrightarrow{\varphi} & E_1 \times E_2 \\ & \searrow \pi & \downarrow \psi \\ & & S \end{array}$$

commutes and ψ^* is a Hodge isomorphism. Indeed the surface $E_1 \times E_2$ is an Abelian surface and in particular it has Kodaira dimension $k(E_1 \times E_2) = 0$. Conversely S is a surface of general type, i.e. $k(S) = 2$, and then ψ does not exist.

Remark 2.5.2. In general the Theorem also does not imply the existence of intermediate covering of the curves C, D . More precisely there are no subgroups H_C, H_D of G such that $C/H_C \cong E_1, C/H_D \cong E_2$ where E_1, E_2 are elliptic curves such that $H^2(S, \mathbb{Q}) \cong H^2(E_1 \times E_2, \mathbb{Q})$, see the following example.

Example 2.5.1. Consider again the unmixed ramification structure studied in Example 2.3.1 and in Section 2.4.2.

We summarize what we need. Let G be the abelian group $(\mathbb{Z}_3)^2$ and let T_D be the spherical system of generators

$$T_D = [(0, 2), (0, 1), (1, 0), (2, 0)].$$

such that the corresponding curve D has genus 4. Consider all the 6 subgroups of G . By [Magma] we easily verify that for all subgroups $H < G$ the quotient curve D/H has genus 0, 2 or 4. In particular there is not any subgroup H such that D/H is an elliptic curve.

2.5.1 About the Picard number

Let S be a surfaces isogenous to a higher product of unmixed type with $\chi(\mathcal{O}_S) = 2$ and $q = 0$. We can compute the Picard number of S using Theorem 2.5.1: let E_1 and E_2 be the elliptic curves such that $H^2(E_1 \times E_2, \mathbb{Q}) \cong H^2(S, \mathbb{Q})$. Then we have $\rho(S) = \rho(E_1 \times E_2)$, where we denote by ρ the Picard number of a given surface.

First of all we notice that $2 \leq \rho(S) \leq 4$:

- the Picard number is upper bounded by $h^{1,1}(S) = 4$;
- the subgroup $(H^2(C, \mathbb{Z}) \otimes H^0(D, \mathbb{Z})) \oplus (H^0(C, \mathbb{Z}) \otimes H^2(D, \mathbb{Z}))$ is contained in the Neron-Severi group and it has dimension 2.

We have the following case:

$$\rho(E_1 \times E_2) = \begin{cases} 4 & \text{if } E \sim E_1 \sim E_2 \text{ has complex multiplication,} \\ 3 & \text{if } E_1 \sim E_2 \text{ but they do not have CM,} \\ 2 & \text{otherwise.} \end{cases}$$

A surface S is said to be a surface with maximal Picard number if $\rho(S) = h^{1,1}(S)$: this kind of surfaces are studied in the recent work of Beauville [Bea14] where a lot of examples are constructed.

The surfaces studied in Sections 2.4.1 and 2.4.3 are new examples of surfaces with maximal Picard number.

Chapter 3

Surfaces with $p_g = q = 2$

In this final chapter we collect some results we obtained in collaboration with J. Commelin and M. Penegini about surfaces S with $p_g(S) = q(S) = 2$.

3.1 Surfaces isogenous to a product with $p_g = q = 2$

Let $S \cong \frac{C \times D}{G}$ be a surface isogenous to a product of unmixed type with $p_g(S) = q(S) = 2$. By Proposition 2.1.9 we distinguish two different cases:

- $g\left(\frac{C}{G}\right) = 2$ and $g\left(\frac{D}{G}\right) = 0$;
- $g\left(\frac{C}{G}\right) = g\left(\frac{D}{G}\right) = 1$.

The surfaces of the first type are called generalized hyperelliptic. We denote by $\alpha : S \rightarrow \text{Alb}(S)$ the Albanese map.

Theorem 3.1.1 ([Pen11]). *Let $S \cong \frac{C \times D}{G}$ be a surface isogenous to a higher product of unmixed type with $p_g(S) = q(S) = 2$. Then $\dim(\alpha(S)) = 1$ if and only if S is generalized hyperelliptic.*

Proposition 3.1.2. *Let S be a surface isogenous to a higher product with $p_g(S) = q(S) = 2$. Then the Hodge diamond is fixed:*

$$\begin{array}{ccccc}
 & & 1 & & \\
 & & 2 & & 2 \\
 2 & & 6 & & 2 \\
 & & 2 & & 2 \\
 & & 1 & &
 \end{array} \tag{3.1}$$

Proof. By hypothesis we have $h^{2,0}(S) = 2$, $h^{1,0}(S) = 2$ and so $\chi(\mathcal{O}_S) = 1$. By Proposition 2.1.7 we obtain $e(S) = 4$ and

$$h^{1,1}(S) = e(S) - 2 + 4q(S) - 2p_g(S) = 6.$$

□

From now on we focus exclusively on surfaces S isogenous to a higher product of Albanese generak type, i.e. surfaces that are not generalized hyperelliptic.

Let us consider the second cohomology group $H^2(S, \mathbb{Q})$. We have the decomposition $H^2(S, \mathbb{Q}) \cong U \oplus Z$, by Proposition 2.3.3. In this case we can further decompose the Hodge structure Z as $Z \cong Z_1 \oplus Z_2$:

$$\begin{aligned} Z_1 &:= H^1(C, \mathbb{Q})^G \otimes H^1(D, \mathbb{Q})^G, \\ Z_2 &:= \bigoplus_{\chi \in \hat{G} - \chi_1} (H^1(C, \mathbb{Q}) \otimes H^1(D, \mathbb{Q}))^\chi, \end{aligned}$$

where \hat{G} is the set of the irreducible characters of G and χ_1 is the trivial character.

Remark 3.1.1. This further decomposition of $H^2(S, \mathbb{Q})$ is trivial in the case studied in Chapter 2. Indeed if $q(S) = 0$ we get $\dim Z_1 = 0$ and $Z = Z_2$.

As in the previous Chapter, the substructure U has dimension 2 and it is isomorphic to $\mathbb{Q}^2(-1)$.

Consider the elliptic curves $E_C := C/G$ and $E_D := D/G$: we observe that $H^1(E_C, \mathbb{Q}) \cong H^1(C, \mathbb{Q})^G$ and $H^1(E_D, \mathbb{Q}) \cong H^1(D, \mathbb{Q})^G$ by Lemma 2.1.8. Then Z_1 has dimension 4 and we get

$$Z_1 \oplus U \cong H^2(E_C \times E_D, \mathbb{Q}).$$

Surfaces $S \cong \frac{C \times D}{G}$ isogenous to a higher product of unmixed type with $p_g = q = 2$ have been classified in [Pen11]. If we assume that S is not generalized hyperelliptic only three groups occur. They are listed in Table 3.1.

Analyzing the three families of surfaces we obtain:

Theorem 3.1.3. *Let S be a surface isogenous to a higher product of unmixed type with $p_g = q = 2$ and not generalized hyperelliptic. Then there exist elliptic curves E_C, E_D, L_C, L_D such that*

$$H^2(S, \mathbb{Q}) \cong (H^2(E_C \times E_D, \mathbb{Q})) \oplus (H^1(L_C, \mathbb{Q}) \otimes H^1(L_D, \mathbb{Q})),$$

as rational Hodge structures.

We fix some notation that will be useful in the rest of this Section. As in Chapter 2 we denote by $f : C \rightarrow C/G$ and $h : D \rightarrow D/G$ the covering maps induced on C and D by the action of G . Since C/G and D/G are elliptic curves the maps f and h are not determined by a spherical system of generators. However, by the Riemann existence Theorem, they are still determined by the base curve (in this case an elliptic curve), the branch points p_1, \dots, p_r and the elements $S_G(p_i)$ where $S_G(p_i)$ is an element of G that acts trivially on the preimage of p_i and such that the cardinality of the

G	$ G $	SGL	$g(C)$	$g(D)$
\mathcal{D}_4	8	$\langle 8, 3 \rangle$	3	5
\mathcal{S}_3	6	$\langle 6, 1 \rangle$	3	4
$\mathbb{Z}_2 \times \mathbb{Z}_2$	4	$\langle 4, 2 \rangle$	3	3

Table 3.1: List of surface isogenous to a higher product of unmixed type with $p_g = q = 2$ not generalized hyperelliptic. SGL is the pair that identifies the group in the Small Groups Library.

preimage is $\#G/\text{ord}(g)$. In other words if $p_i \in C/G$ is a branch point of $f : C \rightarrow C/G$ we have:

$$\{g \in G : g \cdot q = q \ \forall q \in f^{-1}(p_i)\} = \langle S_G(p_i) \rangle. \quad (3.2)$$

We denote by φ_C and φ_D the representations of G induced on the first cohomology groups of C and D . The Broughton formula holds also in this case (see [Bro87] for the general formulation).

Remark 3.1.2. In the following we describe the three families of surfaces isogenous to a higher product, point out that the Theorem 3.1.3 holds for all of them. We do not prove that such surfaces exist: for this please refer to [Pen11].

Remark 3.1.3. In order to construct the elliptic curves L_C and L_D we use the group algebra decomposition, explained in Section 2.2.2. An equivalent construction can be done using Jacobians and Prym varieties.

3.1.1 The case $G = \mathcal{D}_4$

Let G be the dihedral group \mathcal{D}_4 :

$$\mathcal{D}_4 = \langle r, s \mid r^4 = s^2 = (rs)^2 = 1 \rangle.$$

There exist C, D curves of genus $g(C) = 3$ and $g(D) = 5$ such that the group G acts on both curves and

- the covering map $f : C \rightarrow C/G \cong E_C$ has one branch point fixed by r^2 ;
- the covering map $h : D \rightarrow D/G \cong E_D$ has two branch points fixed by s and r^2s .

Consider the character tabel of G :

D_4	Id	r^2	rs	s	r
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

G has 5 irreducible self-dual complex representations, all of them with Schur index 1; so it has 5 absolutely irreducible rational representations τ_1, \dots, τ_5 where τ_i is the representation associated to the character χ_i , $i = 1, \dots, 5$. We decompose φ_C and φ_D by the Broughton formula:

	χ_1	χ_2	χ_3	χ_4	χ_5
φ_C	2	0	0	0	2
φ_D	2	0	2	2	2

Let A_{C_5} and A_{D_5} be the isotypical components related to τ_5 in $H^1(C, \mathbb{Q})$ and $H^1(D, \mathbb{Q})$ and notice that $\dim A_{C_5} = \dim A_{D_5} = 4$. The action of the element s provides decompositions $A_{C_5} = B_{C_5}^{\oplus 2}$ and $A_{D_5} = B_{D_5}^{\oplus 2}$, where $\dim B_{C_5} = \dim B_{D_5} = 2$. Using the same idea of the proof of Theorem 2.3.8 we get $Z_2 \cong B_{C_5} \otimes B_{D_5}$. By remark 2.2.5 there exist elliptic curves L_C, L_D such that $B_{C_5} \cong H^1(L_C, \mathbb{Q})$ and $B_{D_5} \cong H^1(L_D, \mathbb{Q})$.

3.1.2 The case $G = \mathcal{S}_3$

This case is very similar to the previous one. Let G be the symmetric group \mathcal{S}_3 . There exist C, D curves of genus $g(C) = 3$ and $g(D) = 4$ such that the group G acts on both curves and

- the covering map $f : C \rightarrow C/G \cong E_C$ has one branch point fixed by $(1, 2)$;
- the covering map $h : D \rightarrow D/G \cong E_D$ has two branch points fixed by $(1, 2, 3)$.

The group G has 3 irreducible self-dual complex representations with Schur index 1:

\mathcal{S}_3	Id	$[(1, 2)]$	$[(1, 2, 3)]$
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

It follows that G has exactly 3 absolutely irreducible rational representations τ_1, \dots, τ_3 associated to the characters χ_1, \dots, χ_3 . By the Broughton formula we get:

	χ_1	χ_2	χ_3
φ_C	2	0	2
φ_D	2	2	2

Now we conclude as in the \mathcal{D}_4 case: first we decompose into isotypical components, and then we use the element $(1, 2)$ to further decompose isotypical components related to the representation of dimension 2.

3.1.3 The case $G = \mathbb{Z}_2 \times \mathbb{Z}_2$

Let G be the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$. There exist C, D curves of genus $g(C) = g(D) = 3$ such that the group G acts on both curves and

- the covering map $f : C \rightarrow C/G \cong E_C$ has two branch point fixed by $(1, 0)$;
- the covering map $h : D \rightarrow D/G \cong E_D$ has two branch points fixed by $(0, 1)$.

Since G is abelian it has exactly 4 absolutely irreducible rational representations of dimension 1. We denote by τ_i the irreducible rational representation related to χ_i .

	Id	$(0, 1)$	$(1, 0)$	$(1, 1)$
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

By the Broughton formula we get

	χ_1	χ_2	χ_3	χ_4
φ_C	1	1	0	1
φ_D	1	0	1	1

Let us consider the isotopical components A_{C_4}, A_{D_4} of $H^1(C, \mathbb{Q})$ and $H^1(D, \mathbb{Q})$ related to the irreducible rational representation τ_4 . They are Hodge sub-structure of dimension 2. We conclude as above.

Appendix A

More results

A.1 Curves of genus two with an automorphism of order 3

Let C be a curve of genus 2 with an automorphism $\eta : C \rightarrow C$ of order three such that $C/\langle\eta\rangle \cong \mathbb{P}^1$. We denote by $\pi : C \rightarrow \mathbb{P}^1$ the covering map given by the action of $\mathbb{Z}_3 \cong \langle\eta\rangle$.

The morphism π has four branch points, by Riemann-Hurwitz formula. Then a model for the curve is:

$$y^3 = (x - a_1)^{n_1}(x - a_2)^{n_2}(x - a_3)^{n_3}(x - a_4)^{n_4},$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C}$ are distinct values and the covering map is given by the projection over x . Notice that:

- $\forall i \in \{1, 2, 3, 4\} \ n_i \in \{1, 2\}$, otherwise we set $\tilde{y} = y/(x - a_i)$;
- $\sum n_i \equiv 0 \pmod{3}$, otherwise π ramifies over $\infty \in \mathbb{P}^1$;

We obtain

$$y^3 = (x - a_1)(x - a_2)(x - a_3)^2(x - a_4)^2. \quad (\text{A.1})$$

Up to compose π with a Möbius transformation, we can assume $a_1 = 0$, $a_2 = 1$ and $a_3 = \infty$: then C is determined by the choice of a_4 , by Proposition 2.1.14. It follows that this family of curves has dimension 1.

Lemma A.1.1. *Let C be a curve as above. Then there exists $\rho \in \mathbb{C}$ such that C is isomorphic to*

$$v^2 = (u^3 + 1)(u^3 + \rho^6) \quad (\text{A.2})$$

Proof. Consider the model of the curve C given by equation (A.1). The space of the holomorphic 1-forms is generated by

$$H^0(C, K_C) = \left\langle \frac{dx}{y}, \frac{(x - a_3)(x - a_4)dx}{y^2} \right\rangle.$$

The curve C has genus 2 and so it is hyperelliptic. It follows that the map $f : C \rightarrow \mathbb{P}^1$ given by

$$f(x, y) = \frac{(x - a_3)(x - a_4)}{y}$$

is a map of degree 2. Let us consider x , y , and f as elements in $\mathbb{C}(C)$, the rational functions field of C . We obtain

$$f^3 = \frac{(x - a_3)(x - a_4)}{(x - a_1)(x - a_2)}.$$

It follows that $\mathbb{C}(C) \cong \mathbb{C}(x, y) \cong \mathbb{C}(f, x)$. We define $z \in \mathbb{C}(C)$ by

$$z := \frac{2(1 - f^3)x + (a_1 + a_2)f^3 - (a_3 + a_4)}{a_1 - a_2}.$$

As before we get $\mathbb{C}(C) \cong \mathbb{C}(f, z)$ and f , z are related by the equation:

$$z^2 = f^6 + \frac{4a_1a_2 + 4a_3a_4 - 2(a_1 + a_2)(a_3 + a_4)}{(a_1 - a_2)^2}f^3 + \frac{(a_3 - a_4)^2}{(a_1 - a_2)^2}.$$

Thus there exist α and β , dependent on a_1 , a_2 , a_3 and a_4 , such that

$$z^2 = (f^3 + \alpha)(f^3 + \beta).$$

In order to conclude we set

$$u := \frac{f}{\sqrt[3]{\alpha}}, \quad v := \frac{z}{\alpha}, \quad \rho^6 := \frac{\beta}{\alpha}.$$

□

Lemma A.1.2. *Let C be a curve in the family described above. Then it admits an involution $\sigma : C \rightarrow C$ such that $C/\langle \sigma \rangle$ is an elliptic curve.*

Proof. Consider the model of the curve C given by equation (A.2). We define the map σ setting

$$\sigma(u, v) = \left(\frac{\rho^2}{u}, \frac{v\rho^3}{u^3} \right).$$

First of all we verify that $\sigma(C) \subseteq C$:

$$\left(\frac{\rho^6}{u^3} + 1 \right) \left(\frac{\rho^6}{u^3} + \rho^6 \right) = \frac{\rho^6}{u^6}(\rho^6 + u^3)(1 + u^3) = \frac{\rho^6}{u^6}v^2.$$

The map $\sigma : C \rightarrow C$ is an involution:

$$\sigma^2(u, v) = \sigma \left(\frac{\rho^2}{u}, \frac{v\rho^3}{u^3} \right) = \left(\rho^2 \frac{u}{\rho^2}, \frac{u\rho^3}{u^3} \rho^3 \frac{u^3}{\rho^6} \right) = (u, v).$$

We notice that (u, v) is a fixed point for σ if and only if $u^2 = \rho^2$ and $u^3 = \rho^3$, i.e. if and only if $u = \rho$. It follows that $\sigma : C \rightarrow C$ has two fixed points and then $C/\langle \sigma \rangle$ is an elliptic curve by the Riemann-Hurwitz formula. □

Corollary A.1.2.1. *The Jacobian $Jac(C)$ is not simple. In particular there exists an elliptic curve E such that $Jac(C) \sim E \times E$.*

Proof. The automorphism σ induces a decomposition $Jac(C) \sim E_1 \times E_2$. However the two elliptic curves are not fixed by the automorphism η of order three, then it provides an isogeny $E_1 \sim E_2$. \square

We can explicitly compute the elliptic curve $E = C/\langle\sigma\rangle$. Consider $g, h \in \mathbb{C}(C) \cong \mathbb{C}(u, v)$:

$$\begin{aligned} g &:= v \left(\frac{1}{\rho u} + \frac{1}{u^2} \right), \\ h &:= u + \frac{\rho^2}{u}. \end{aligned}$$

They are invariant over the action of σ :

$$\begin{aligned} g \circ \sigma(u, v) &= g \left(\frac{\rho^2}{u}, \frac{v\rho^3}{u^3} \right) = \frac{v\rho^3}{u^3} \left(\frac{u}{\rho^3} + \frac{u^2}{\rho^4} \right) = g(u, v), \\ h \circ \sigma(u, v) &= h \left(\frac{\rho^2}{u}, \frac{v\rho^3}{u^3} \right) = \frac{\rho^2}{u} + \rho^2 \frac{u}{\rho^4} = h(u, v). \end{aligned}$$

Then g and h define two rational functions on E and they are related by the equation:

$$g^2 = \frac{h^4}{\rho^2} + \frac{2}{\rho}h^3 - 3h^2 + \left(\rho^4 + \frac{1}{\rho^2} - 6\rho \right)h + 2\rho^5 + \frac{2}{\rho}. \quad (\text{A.3})$$

Consider the following chain of inclusions:

$$\mathbb{C}(C) \cong \mathbb{C}(u, v) \supset \mathbb{C}(E) \supseteq \mathbb{C}(g, h).$$

From $u^2 - uh + \rho^2 = 0$ follows that

$$[\mathbb{C}(u, v) : \mathbb{C}(g, h)] = 2.$$

The curves C and E have different genus and then $\mathbb{C}(C) \not\cong \mathbb{C}(E)$. Thus we conclude that $\mathbb{C}(E) \cong \mathbb{C}(g, h)$: in particular E is defined by the equation (A.3).

Remark A.1.1. We can compute explicitly the j -invariant of the elliptic curve E from the equation (A.3). Since $j(E)$ is a function of ρ , and then it is not a fixed value, we prove that the elliptic curve E changes when we move C in the family.

A.2 Dessin d'enfants and the quaternion group

Let C be a curve of genus 2 and let $G < \text{Aut}(C)$ be a group of automorphisms isomorphic to the quaternion group Q_8 . From now on we identify G with Q_8 and we assume the notation of Example 2.2.2:

$$Q_8 = \langle -1, i, j, k \mid (-1)^2 = 1, i^2 = j^2 = k^2 = ijk = -1 \rangle.$$

Lemma A.2.1. *The quotient curve C/G is isomorphic to \mathbb{P}^1 . In particular the covering map*

$$\pi : C \rightarrow C/G \cong \mathbb{P}^1$$

is determined by the spherical system of generators $[i, j, -k]$ of type $[4^3]$.

Proof. The cases $g(C/G) = 2$ and $g(C/G) = 1$ are excluded by the Riemann-Hurwitz formula. Let assume $C/G \cong \mathbb{P}^1$: by Riemann-Hurwitz formula we get:

$$2 = -16 + \sum_{x \in C} (m_x(\pi) - 1).$$

It follows that $\pi : C \rightarrow \mathbb{P}^1$ has three branch points, all of them with multiplicity 4. This means that the spherical system of generators associated to π has type $[4^3]$. From the structure of Q_8 we deduce that all the spherical systems of generators of this type are isomorphic to $[i, j, -k]$. \square

Corollary A.2.1.1. *The curve C is unique, up to isomorphisms.*

Proof. Up to compose with a Möbius transformation, we can assume that $\pi : C \rightarrow \mathbb{P}^1$ ramifies over $\{0, 1, \infty\}$. So, by the Riemann Existence Theorem (Proposition 2.1.14), C is unique. \square

Lemma A.2.2. *A model for (C, π) is*

$$C : \{y^2 = x(x^4 - 1)\}, \quad \pi(x, y) = -\frac{1}{4} \frac{(x^2 - 1)^2}{x^2}. \quad (\text{A.4})$$

Proof. The genus 2 curve defined by $y^2 = x(x^4 - 1)$ admits an automorphism group isomorphic to Q_8 , by [BL04, p.340]. By uniqueness (Corollary A.2.1.1) this curve is isomorphic to C .

Let us consider the normal subgroup $\mathbb{Z}_2 \triangleleft Q_8$ given by $\mathbb{Z}_2 \cong \langle -1 \rangle$. We observe that:

- $C/\mathbb{Z}_2 \cong \mathbb{P}^1$ by the Riemann-Hurwitz formula, and in particular -1 is the hyperelliptic involution on C ;
- $Q_8/\mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is the Klein group.

We get the following covering diagram:

$$\begin{array}{ccc} C & \xrightarrow[2:1]{q} & \mathbb{P}^1 \\ & \searrow \pi & \downarrow p \\ & & \mathbb{P}^1 \end{array} \quad \begin{array}{c} \\ \\ 4:1 \end{array}$$

The hyperelliptic involution on $C : \{y^2 = x(x^4 - 1)\}$ is given by

$$q : C \rightarrow \mathbb{P}^1, \quad q(x, y) \mapsto x.$$

With some extra computations we can also realize that the map p is given by

$$p : \mathbb{P}^1 \rightarrow \mathbb{P}^1, \quad p(x) = -\frac{1}{4} \frac{(x^2 - 1)^2}{x^2}.$$

By construction $\pi = p \circ q$ and so

$$\pi : C \rightarrow \mathbb{P}^1, \quad \pi(x, y) = -\frac{1}{4} \frac{(x^2 - 1)^2}{x^2}.$$

Notice that π ramifies over $\{0, 1, \infty\}$:

$$\begin{aligned} \pi^{-1}(0) &= \{(1, 0), (-1, 0)\}, \\ \pi^{-1}(1) &= \{(i, 0), (-i, 0)\}, \\ \pi^{-1}(\infty) &= \{(0, 0), \infty\}. \end{aligned}$$

□

Lemma A.2.3. *Let C be the curve of genus 2 described above. Then there exists an involution $\sigma \in \text{Aut}(C)$ such that $C/\langle \sigma \rangle$ is an elliptic curve.*

Proof. Consider the model of the curve C given by equation (A.4). We define σ by

$$\sigma(x, y) = \left(\frac{-x+1}{x+1}, \frac{2\sqrt{2}y}{(x+1)^3} \right).$$

First of all we verify that $\sigma^2(x, y) = (x, y)$:

$$\begin{aligned} \sigma^2(x, y) &= \sigma \left(\frac{-x+1}{x+1}, \frac{2\sqrt{2}y}{(x+1)^3} \right) = \\ &= \left(\left[\left(\frac{x-1}{x+1} \right) + 1 \right] \left(\frac{x+1}{2} \right), \left(\frac{8y}{(x+1)^3} \right) \left(\frac{x+1}{2} \right)^3 \right) = \\ &= \left(\frac{2x}{x+1} \frac{x+1}{2}, \frac{8y}{(x+1)^3} \frac{(x+1)^3}{8} \right) = (x, y). \end{aligned}$$

Now we prove that $\sigma(C) \subseteq C$:

$$\begin{aligned} \frac{(2\sqrt{2}y)^2}{(x+1)^6} - \frac{-x+1}{x+1} \left(\left(\frac{-x+1}{x+1} \right)^4 - 1 \right) &= \frac{8y^2}{(x+1)^6} - \frac{8x(x-1)(x^2+1)}{(x+1)^5} = \\ &= \frac{8}{(x+1)^6} (y^2 - x(x-1)(x+1)(x^2+1)) = \frac{8(y^2 - x(x^4-1))}{(x+1)^6} = 0 \end{aligned}$$

A point (x, y) is a fixed point for σ if and only if:

$$\begin{cases} x = \frac{-x+1}{x+1} \\ y = \frac{2\sqrt{2}y}{(x+1)^3} \end{cases}, \quad \begin{cases} x = -1 \pm \sqrt{2} \\ (\pm\sqrt{2})^3 = 2\sqrt{2} \end{cases}.$$

We get two solutions in C :

$$Fix_\sigma(C) = \left\{ \left(-1 + \sqrt{2}, \sqrt{-1 + \sqrt{2}} \right), \left(-1 + \sqrt{2}, -\sqrt{-1 + \sqrt{2}} \right) \right\}.$$

Then $C/\langle\sigma\rangle$ is an elliptic curve, by the Riemann-Hurwitz formula. \square

Remark A.2.1. The unique involution in Q_8 is the hyperelliptic involution -1 . Then the automorphism group $Aut(C)$ is strictly bigger than Q_8 .

Corollary A.2.3.1. *The Jacobian $Jac(C)$ is not simple. In particular we get $Jac(C) \sim E \times E$.*

Proof. The involution σ induces a decomposition $Jac(C) \sim E_1 \times E_2$. However the two elliptic curves are not fixed by the action of the group Q_8 ; so at least one element of the group provides an isogeny $E_1 \sim E_2$ \square

Lemma A.2.4. *The curve $C/\langle\sigma\rangle$ is the elliptic curve*

$$E_{\sqrt{-2}} = \frac{\mathbb{C}}{\mathbb{Z} \oplus \sqrt{-2}\mathbb{Z}}.$$

Proof. We follow the proof of [Bea14, Example 2]. Consider the elliptic curve $\tilde{E} : \{v^2 = u(u+1)(u-2\alpha)\}$, with $\alpha = 1 - \sqrt{2}$. There is a map from C to \tilde{E} given by

$$(x, y) \mapsto \left(\frac{x^2+1}{x-1}, \frac{y(x-\alpha)}{(x-1)^2} \right).$$

It follows $(C/\langle\sigma\rangle) \sim \tilde{E}$ and $Jac(C) \sim \tilde{E} \times \tilde{E}$. The j -invariant of \tilde{E} is 8000, so \tilde{E} is the elliptic curve $E_{\sqrt{-2}}$. \square

We conclude this section with a brief digression on the Grothendieck theory of the dessin d'enfants. We assume here the notations of [GGD12]. The pair (C, π) defined by equation (A.4) is a Belyi pair. Then, by Grothendieck correspondence [GGD12, Chapter 3], we associate to (C, π) a dessin

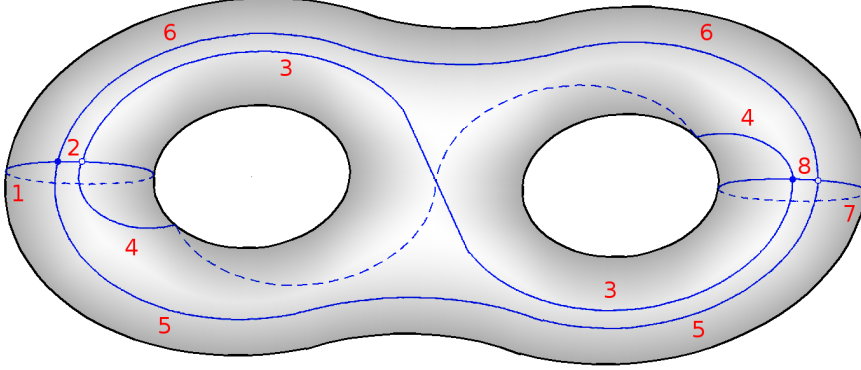


Figure A.1: The dessin d'enfants (\tilde{C}, f)

d'enfants. Notice that by Corollary A.2.1.1 (C, π) is the unique Belyi pair with $g(C) = 2$ and such that $\pi : C \rightarrow \mathbb{P}^1$ is induced by the action of Q_8 .

Let us consider the dessin d'enfants in Figure A.1 and we denote by (\tilde{C}, f) the corresponding Belyi pair. We immediately notice that $g(\tilde{C}) = 2$. Any dessin is determined by its permutation representation [GGD12, Chapter 4]. For (\tilde{C}, f) we get:

$$\begin{aligned}\sigma_0 &= (1, 3, 2, 4)(5, 7, 6, 8); \\ \sigma_1 &= (1, 5, 2, 6)(3, 8, 4, 7).\end{aligned}$$

In particular we notice that $\sigma_0\sigma_1 = (\sigma_1\sigma_0)^{-1}$:

$$\begin{aligned}\sigma_0\sigma_1 &= (1, 7, 2, 8)(3, 5, 4, 6) \\ \sigma_1\sigma_0 &= (1, 8, 2, 7)(3, 6, 4, 5)\end{aligned}$$

It follows that $\langle \sigma_0, \sigma_1 \rangle \cong Q_8$ and then the Belyi pair (\tilde{C}, f) is defined by the action of the quaternion group Q_8 on a curve of genus 2. By Lemma A.2.1, the Belyi pairs (\tilde{C}, f) and (C, π) are isomorphic and in particular the dessin in Figure A.1 is the dessin of (C, π) .

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